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Physics for Biomedical Engineers

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1. Measurements

Physics is based on measurements of physical quantities. Each quantity is a product of number and unit, but independent of the choice of the unit, which is a man-made definition.

Quantity = Value × Unit *Measurement*

1.1. The International System of Units (SI)

As the figure depicts, all things in classical physics can be expressed in terms of the seven base quantities, standardized in the Système International d'unités (SI) system. The MKS system refers to meter, kilogram, and second, while the CGS system uses centimeter, gram and second.

Base quantity	Name	Symbol
length	meter	m
time	second	s
mass	kilogram	kg
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

The figure shows 20 prefixes, which are also standardized in the SI system.

Factor	Name	Symbol	Factor	Name	Symbol
10^{24}	yotta	Y	10^{-1}	deci	d
10^{21}	zetta	Z	10^{-2}	centi	c
10^{18}	exa	E	10^{-3}	milli	m
10^{15}	peta	P	10^{-6}	micro	µ
10^{12}	tera	T	10^{-9}	nano	n
10^{9}	giga	G	10^{-12}	pico	p
10^{6}	mega	M	10^{-15}	femto	f
10^{3}	kilo	k	10^{-18}	atto	a
10^{2}	hecto	h	10^{-21}	zepto	z
10^{1}	deka	da	10^{-24}	yocto	y

All other quantities are derived from the base quantities. For example, velocity (speed) is defined as distance (length) per time. Frequently used units of velocity are kilometre per hour or miles per hour. The SI unit is meter per second (m/s). Acceleration is defined as speed per time, and force is mass times acceleration. Its SI unit is newton = kilogram times meter per second (N = kg · m/s).

The following table provides some examples.

Physical quantity	Unit name	Symbol	Definition of SI unit	Equivalent forms of SI unit
energy work	joule	J	$m^2 \cdot kg \cdot s^{-2}$	$N \cdot m$
force	newton	N	$m \cdot kg \cdot s^{-2}$	$J \cdot m^{-1}$
pressure stress	pascal	Pa	$m^{-1} \cdot kg \cdot s^{-2}$	$N \cdot m^2$
power	watt	W	$m^2 \cdot kg \cdot s^{-3}$	$J \cdot s^{-1}$
electric charge	coulomb	C	$s \cdot A$	$s \cdot A$
electric potential difference	volt	V	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-1}$	$W \cdot A^{-1}$
electric resistance	ohm	Ω	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-2}$	$V \cdot A^{-1}$
electric conductance	siemens	S	$m^{-2} \cdot kg^{-1} \cdot s^3 \cdot A^2$	$A \cdot V^{-1}$
electric capacitance	farad	F	$m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2$	$C \cdot V^{-1}$
magnetic flux	weber	Wb	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-1}$	$V \cdot s$
magnetic flux density	tesla	T	$kg \cdot s^{-2} \cdot A^{-1}$	$Wb \cdot m^{-2}$
inductance	henry	H	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-2}$	$Wb \cdot A^{-1}$
luminous flux	lumen	lm	$m^2 \cdot m^{-2} \cdot cd = cd$	$cd \cdot sr$
illuminance	lux	lx	$m^2 \cdot m^{-4} \cdot cd = m^{-2} \cdot cd$	$lm \cdot m^{-2}$
frequency	hertz	Hz	s^{-1}	s^{-1}
activity of radionuclides	becquerel (replaces currie)	Bq	s^{-1}	s^{-1}
absorbed dose (of ionizing radiation)	gray (replaces red)	Gy	$m^2 \cdot s^{-2}$	$J \cdot kg^{-1}$

1.2. Conversion of Units

Units of the same quantity can be converted into each other.

The following table provides some examples.

Quantity	Equivalent Values
Mass	1 kg = 1000 g = 0.001 metric ton = 2.20462 lb_m = 35.27392 oz
	1 lb_m = 16 oz = 5×10^{-4} ton = 453.593 g = 0.453593 kg
Length	1 m = 100 cm = 1000 mm = 10^6 μm = 10^{10} Å
	1 m = 39.37 in = 3.2808 ft = 1.0936 yd = 0.0006214 mile
	1 ft = 12 in = 1/3 yd = 0.3048 m = 30.48 cm
Volume	1 m^3 = 1000 liters = 10^6 cm^3 = 10^6 ml
	1 m^3 = 35.3145 ft^3 = 220.83 imperial gallons = 264.17 gal = 1056.68 qt
	1 ft^3 = 1728 in^3 = 7.4805 gal = 0.028317 m^3 = 28.317 liters = 28317 cm^3
Force	1 N = 1 kg·m/s^2 = 10^5 dynes = 10^5 g·cm/s^2 = 0.22481 lb_f
	1 lb_f = 32.174 lb_m·ft/s^2 = 4.4482 N
Pressure	1 atm = 1.01325×10^5 N/m^2 (Pa) = 101.325 kPa = 1.01325 bars
	1 atm = 1.01325×10^6 dynes/cm^2
	1 atm = 760 mmHg at 0°C (torr) = 10.333 m H_2O at 4°C = 14.696 lb_f/in^2 (psi)
	1 atm = 33.9 ft H_20 at 4°C = 29.921 inHg at 0°C
Energy	1 J = 1 N·m = 10^7 ergs = 10^7 dyne·cm = 2.778×10^{-7} kW·h
	1 J = 0.23901 cal = 0.7376 ft-lb_f = 9.486×10^{-4} Btu
Power	1 W = 1 J/s = 1.341×10^{-3} hp

1.3. Measurement errors

All measurements are erroneous. There are principle (systematic) errors inherent to the measurement device (e.g., a ruler was calibrated at room temperature but is then used in Africa or Sibiria). These should be avoided, indicated, and can be corrected. Furthermore, all measures have statistical errors due to some arbitrary and sometimes unknown influences (e.g., the reaction time when using a stop watch). The statistical error is reduced with averaging from repeated measurements.

Quantity = (Value ± Error) × Unit *Erroneous Measurement*

If the error is not indicated, one uses the last decimal. Keep that in mind when using electrical calculators.

For instance, a mass of 10.0 kg equals (10.0 ± 0.1) kg and a velocity v = 100 km / 3.0 h = 33 km/h but not 33.3333 km/h. Since the error is not indicated, the actual range of the measured speed is:

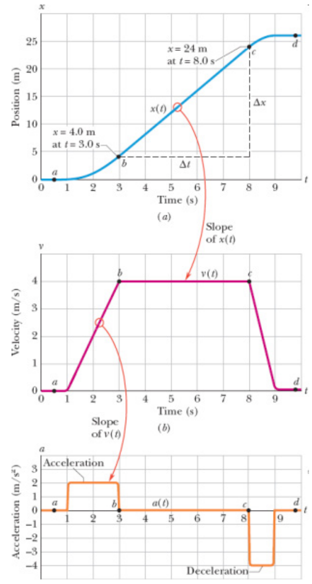
99 km / 3.1 h = 31.9 km/h ≤ v ≤ 34.8 km/h = 101 km / 2.9 h.

Absolute error $x \pm \Delta x$

Relative error $x \pm \frac{\Delta x}{x}$

2. Motion Along Straight Lines

A length measured from an origin (SI base quantity) *Position*



We now consider the movement of a particle (i.e., a point like object or any object that moves as such) along a straight line (linear movement).

2.1. *Position, Displacement, Velocity, and Acceleration*

Smartly, we place our coordinate system along that line of movement. The movement is described completely by the physical quantities: (i) position, (ii) displacement, (iii) velocity, and (iv) acceleration.

The figure graphs these quantities as functions of time. Since motion describes a changing position of our particle, a recording over the time t is used to describe the motion. The uppermost panel describes the position x(t) of our particle.

Frequently, we are not interested in all the intermediate positions of the particle, but only consider a starting $x_1 = x(t_1)$ and an ending point $x_2 = x(t_2)$ of motion (e.g., marks (b) and (c), respectively), disregarding the difference in time t it takes to change between the positions.

Displacement

$$\Delta x = x(t_2) - x(t_1) = x_2 - x_1$$

If we are interested also in the duration, i.e. the interval of time $\Delta t = t_2 - t_1$ that has passed, we define the velocity v.

Velocity (average)

$$v_{\text{avg}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

Velocity (instantanious)

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx(t)}{dt}$$

When a particle's velocity is changed, the particle is said to undergo an acceleration a.

Acceleration (average)

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$$

Acceleration (instantanious)

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x(t)}{dt^2}$$

2.2. *Constant Acceleration*

If the instantaneous acceleration a is constant, it equals the average acceleration. From that, or more general from using the antiderivative of polynomial functions, we obtain:

Displacement (constant acceleration)

$$x = \frac{1}{2}at^2 + v_0 t + x_0$$

Velocity (constant acceleration)

$$v = at + v_0$$

Acceleration (constant acceleration)

$$a = a_{\text{avg}} = \text{const.}$$

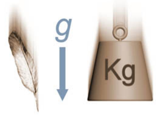
The figure emphasizes that the basic equations for constant acceleration can be used for solving any problem, disregarding the quantity that is missing in the problem statement.

Equation	Missing Quantity
$v = v_0 + at$	$x - x_0$
$x - x_0 = v_0 t + at^2$	v
$v^2 = v_0^2 + 2a(x - x_0)$	t
$x - x_0 = 1/2(v_0 + v)t$	a
$x - x_0 = vt - 1/2at^2$	v_0

2.3. *Free-Fall*

By eliminating the effects of air one would find that the acceleration is independent of the object's characteristics, such as mass, density, or shape. It is the same for all objects.

The figure visualizes a free-fall in vacuum.



Free-fall acceleration is the constant rate of downward acceleration that is found near the Earth's surface. Its magnitude is $g = 9.81\ m/s^2$

Free-Fall Acceleration

Acceleration (free-fall)

$$a = -g = -9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

3. Recap on Mathematics

3.1. *Useful Relations*

Bionomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \ldots$$

Bionomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Factorial

$$n! = \prod_{k=1}^{n} k$$

Quadratic Formula

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Identity

$$(a-b)(a+b) = a^2 - b^2$$

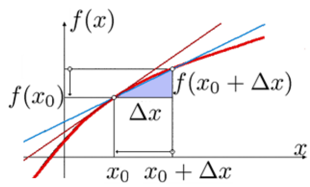
3.2. *Calculus*

A relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output *Function*

Function

$$x \mapsto f(x)$$

The figure visualizes a one-dimensional (1D) function f (red) at point x_0. The tangent at that point is lined in orange. The secant between points x_0 and $x_0 + \Delta x$ is lined blue. The derivation is obtained when Δx becomes very small



Derivation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Derivation (notation)

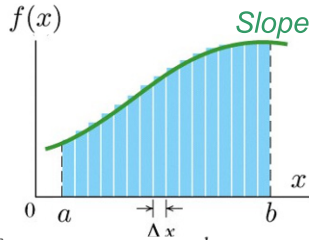
$$f'(x) = f_x = \partial_x f = \frac{\partial f}{\partial x}$$

Chain Rule

$$\frac{\partial}{\partial x}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$$

The derivation of the curve at that point *Slope*

The figure depicts a 1D function f(x). The area under the curve is willed with small rectangles of width Δx. The integral is obtained when Δx becomes very small and hence the number of rectangles infinite.



Integration

$$F(x) = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k = \int_a^b f(x)\, dx$$

Integration (notation)

$$\int_a^b f(x) = F(b) - F(a) = \Big[F(x)\Big]_a^b = F(x)\Big|_a^b$$

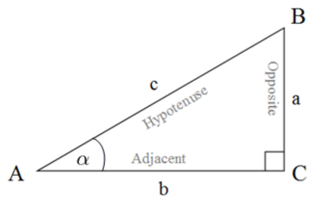
The integration of a curve between to points *Area*

Fundamental Theorem of Calculus

$$F'(x) = f(x) \quad \Leftrightarrow \quad \int f(x)\, dx = F(x)$$

3.3. *Trigonometry*

The figure depicts a right angled triangle of points A, B, and C that is composed of three lines a, b, and c denoting the opposite, adjacent, and hypotenuse, respectively.



Sine

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypothenuse}} = \frac{a}{c}$$

Cosine

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypothenuse}} = \frac{b}{c}$$

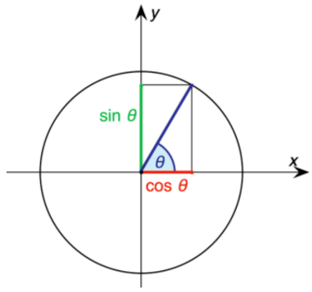
Tangent

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = \frac{\sin(\alpha)}{\cos(\alpha)}$$

Trigonometric Identities

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

The above definitions apply to angles between 0 and 90 degrees (0 and π/2 radians) only. Using the unit circle, one can extend them to all positive and negative arguments.



The figure depicts the unit circle. Combining calculus and inifinite series, the trigonometric functions can be defined for complex numbers. The trigonometric functions are periodic, with a period of 360 degrees or 2π radians.

Euler's Formula

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

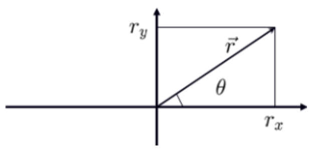
3.4. Vectors

A quantity that has a size and a direction — *Vector*

In one dimension, we define an origin and the sense of direction. The position is one signed number (direction and magnitude). Displacement, velocity, acceleration are also vectors specified just by one signed number.

The figure visualizes a 1D vector.

In two dimensions, again, select an origin and draw two mutually perpendicular lines meeting at that origin. Select directions for horizontal (x) and vertical (y) axes. Any position is given by two signed numbers (vector).



The figure visualizes a two-dimensional (2D) vector.

Length

$$\vec{r} = \sqrt{r_x^2 + r_y^2}$$

Direction

$$\theta = \arctan\left(\frac{r_x}{r_y}\right)$$

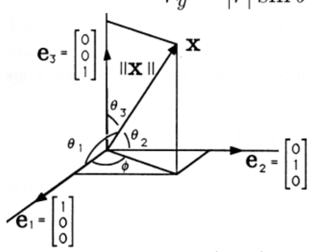
X-Component

$$r_x = |\vec{r}|\cos\theta$$

Y-Component

$$r_y = |\vec{r}|\sin\theta$$

The figure visualizes a three-dimensional (3D) vector. To identify a 3D vector, we either need magnitude and two angles or three components (signed number).



Vector Addition (Communative Law)

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Vector Addition (Associative Law)

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

Vector Subtraction

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

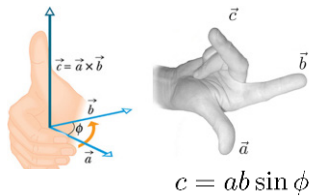
Multiplication (vector and scalar)

$$\vec{b} = s \cdot \vec{a}$$

Scalar Product $\vec{b} \cdot \vec{a} = ab\cos\phi$

Vector Product $\vec{c} = \vec{a} \times \vec{b}$

The figure depicts the right hand rule that can be used to determine the resulting direction of a vector multiplication that yields a vector (cross product, vector product).



Vector Product (magnitude) $c = ab\sin\phi$

Vector Product (parallel) $c = 0$

Vector Product (perpendicular) $\vec{c} = \vec{c}_{\max}$

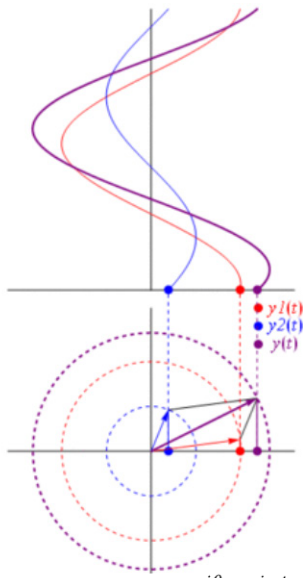
Vector Product (reversed) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

3.5. Phasors

A phasor is a representation of a sinusoidal function whose amplitude A, frequency ω, and phase θ are time-invariant. Phasors separate the dependencies on A, ω, and θ into three independent factors. Hence, trigonometry reduces to algebra, and linear differential equations become algebraic ones.

In essence, a phasor is a vector that has a magnitude equal to the amplitude A. It rotates counter-clockwise around the origin with ω and the offset θ.

The figure illustrates a vector addition of two phasors, representing sinusoidal curves. Phasors easily allow superimposing waves of different amplitudes.



Euler's Formula $A \cdot \cos(\omega t + \theta) = \mathrm{Re}\{Ae^{i\theta} \cdot e^{i\omega t}\}$

Phasor $A\angle\theta = Ae^{i\theta}$

Phasor Addition $A_3\angle\theta_3 = A_1\angle\theta_1 + A_2\angle\theta_2$

Amplitude $A_3 = \sqrt{(A_1\cos\theta_1 + A_2\cos\theta_2)^2 + (A_1\sin\theta_1 + A_2\sin\theta_2)^2}$

Phase $\theta_3 = \arctan\left(\frac{A_1\sin\theta_1 + A_2\sin\theta_2}{A_1\cos\theta_1 + A_2\cos\theta_2}\right)$

3.6. Nabla Notation

The vector differential operator usually is represented by the Nabla (Del) symbol ∇. When applied to a function defined on a one-dimensional domain, it denotes its standard derivative as defined in calculus. When applied to a scalar field f(x,y,z), which is a function defined on a multi-dimensional domain, ∇ may denote the gradient (locally steepest slope) of a scalar field (f,g) or a vector field (u,v), the divergence of a vector field, or the curl (rotation) of a vector field, depending on the way it is applied.

Vector Field $\vec{v}(x,y,z) = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

Nabla Operator (Cartesian coordinate system R^3) $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

The Nabla operator applied to a scalar field yields a vector field.

Gradient
$$\nabla f = \vec{\mathrm{grad}} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

Gradient (multiplication)
$$\nabla(fg) = f\nabla g + g\nabla f$$

Gradient (scalar product)
$$\nabla(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot\nabla)\vec{v} = (\vec{v}\cdot\nabla)\vec{u} = \vec{u}\times(\nabla\times\vec{v}) = \vec{v}\times(\nabla)\times\vec{u})$$

The Nabla operator applied to a vector field yields a scalar field, if the operation is the scalar product.

Divergence
$$\nabla\cdot\vec{v} = \mathrm{div}\ \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Divergence (multiplication)
$$\nabla\cdot(f\vec{v}) = f(\nabla\cdot\vec{v}) + \vec{v}\cdot(\nabla f)$$

Divergence (vector product)
$$\nabla\cdot(\vec{v}\times\vec{u}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v})$$

The Nabla operator applied to a vector field yields a vector field, if the operation is the vector product.

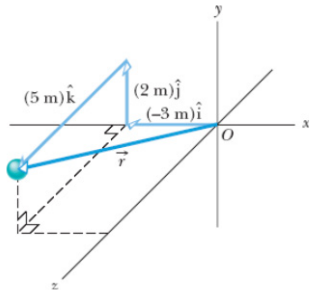
Curl
$$\nabla\times\vec{v} = \vec{\mathrm{curl}}\ \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$
$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

Curl (multiplication)
$$\nabla\times(f\vec{v}) = (\nabla f)\times\vec{v} + f(\nabla\times\vec{v})$$

Curl (vector product)
$$\nabla\times(\vec{v}\times\vec{u}) = \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u}) + (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v}$$

4. Motion in Two and Three Dimensions

Motion in two and three dimension is more complex and not restricted to a line. In fact, all physical quantities we have used to describe the motion are vector quantities, and any 3D motion in space can be regarded as a superimposition of the linear motions along the components of that vectors.

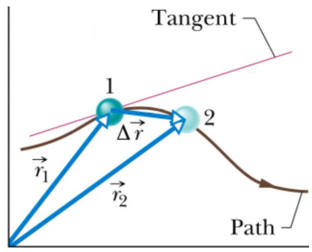


4.1. Position

The figure depicts the position vector r for a particle as the vector sum of its components.

Position $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

4.2. Displacement

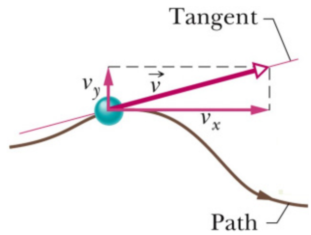


The figure depicts the displacement vector Δr for a particle that has moved from position 1 (indicated by position vector r_1) to position 2 (indicated by position vector r_2).

Displacement $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

Displacement (components) $\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

4.3. Velocity



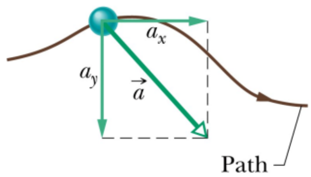
Again, we need to differentiate the average from the instantaneous velocity.

The figure depicts the velocity of a particle, along with its scalar components vector r for a particle as the vector sum of its components.

Velocity $\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t} \quad , \quad \vec{v} = \frac{d\vec{r}}{dt}$

Velocity (components) $v_x = \frac{dx}{dt} \quad , \quad v_y = \frac{dy}{dt} \quad , \quad v_z = \frac{dz}{dt}$

4.4. Acceleration

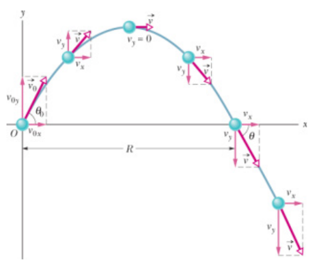


The figure depicts the acceleration vector a for a particle as the vector sum of its components.

Acceleration $\vec{a}_{\text{avg}} = \frac{\Delta\vec{a}}{\Delta t} \quad , \quad \vec{a} = \frac{d\vec{v}}{dt}$

Acceleration (components) $a_x = \frac{dv_x}{dt} \quad , \quad a_y = \frac{dv_y}{dt} \quad , \quad a_z = \frac{dv_z}{dt}$

4.5. Projectile Motion



The projectile motion is a two dimensional motion, where a particle moves in the vertical plane with some initial velocity v_0, and the acceleration is free-fall acceleration g, which is always directed downwards.

The figure depicts the path of a projectile motion. The horizontal range is denoted R. The initial angle is denoted θ_0.

Since horizontal and vertical motion are independent of each other (neither motion affects the other), we analyze that motion separately for its components.

Horizontal Motion

$$x - x_0 = v_{0x}t = (v_0 \cos\theta_0)t$$

Vertical Motion

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

Trajectory (equation of path)

$$y = (\tan\theta_0)x - \frac{gx^2}{2(v_0\cos\theta_0)^2}$$

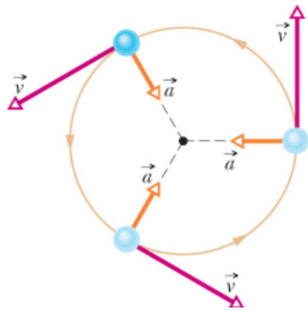
Horizontal Range

$$R = \frac{v_0^2}{g}\sin 2\theta_0$$

4.6. Uniform Circular Motion

A particle travels around a circle are at constant (uniform) speed

Circular Motion



Although the speed does not vary in a uniform circular motion, the moving particle is continuously accelerating.

The figure shows the relationship between the velocity and acceleration vectors at various stages during uniform circular motion.

The velocity is always directed tangent to the circle. The acceleration is always directed radially inward. Because of this, it is called centripetal (center seeking).

Acceleration (centripetal)

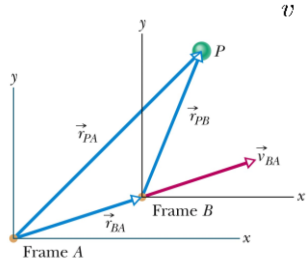
$$a = |\vec{a}| = \frac{v^2}{r}$$

The time needed to travel the circumference of the circle (distance of 2πr) is called the period of revolution T.

Period

$$T = \frac{2\pi r}{v}$$

4.7. Relative Motion



The position r of a particle and its velocity v depend on the reference frame of whoever is observing or measuring the movement.

The figure depicts the position of P measured from a reference frame A (denoted r_{PA}) and the position r_{PB} (measured from frame B). The velocity v_{BA} is measured of reference frame B relative to frame A.

Relative Position

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

Relative Velocity

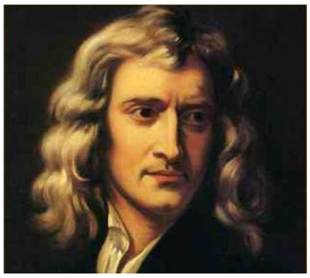
$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

However, observers on different frames of reference that move at a constant velocity relative to each other will measure the same acceleration for a moving particle.

Relative Acceleration

$$\vec{a}_{PA} = \vec{a}_{PB}$$

5. Force and Motion



Isaac Newton (1642–1727) has analyzed the how and why of motion. He found three fundamental laws relating the mass m and the force F with the acceleration a.

5.1. Newton's 1st Law

If no force acts on a body, then the body's velocity cannot change; that is, the body cannot accelerate — *Newton's 1st Law*

Newton's 1st Law

$$\vec{F}_{\mathrm{net}} = 0 \quad \Rightarrow \quad \vec{a} = 0$$

If two or more forces act on a body, we can find their net force F_{net} (resultant force) by adding the individual forces vectorially.

The vectors a and F are defined on a certain reference frame. For instance, when neglecting Earth's astronomical motions, the ground forms a so called intertial frame.

A reference frame in which Newton's Laws hold — *Inertial Frame*

5.2. Newton's 2nd Law

The mass m (SI base quantity, unit kilogram (kg)) is a property of an object that measures how hard it is to change its motion. It determines how much force is exerted or needed attempting to accelerate the body. There is no other physical sensation of mass.

The net force on a body is equal to the product of the body's mass and its acceleration — *Newton's 2nd Law*

Newton's 2nd Law

$$\vec{F}_{\mathrm{net}} = m \cdot \vec{a}$$

$$\vec{a} = 0 \quad \Rightarrow \quad \vec{F}_{\mathrm{net}} = 0$$

The acceleration component along a given axis is caused only by the sum of force components along that same axis.

Newton's 2nd Law (x component)

$$F_{\mathrm{net,x}} = m \cdot a_x$$

Any composition of one or more objects — *System*

Any force from bodies outside the system — *External Force*

Any force from bodies inside the system — *Internal Force*

A system without external forces — *Closed System*

5.3. Newton's 3rd Law

Let us consider some particular forces.

A certain type of pull directed to a second body — *Gravitational Force*

Gravitational Force

$$\vec{F}_g = -F_g \hat{j} = -mg\hat{j} = m\vec{g}$$

Gravitational Force (on Earth)

$$g = 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

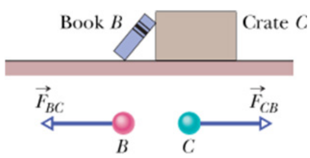
The magnitude of the net force required to prevent a body from falling freely is called weight. Notice that weight is the magnitude of a force.

Weight

$$W = mg$$

Objects in contact exert contact forces. The convention for notation is $F_{A,B}$ or short F_{AB} for the force acting on A due to B.

The figure depicts the contact forces between a book and a crate.



When two bodies interact, the forces on the bodies from each other are always equal in magnitude and opposite in direction

Newton's 3rd Law

Newton's 3rd Law

$$\vec{F}_{BC} = -\vec{F}_{CB}$$

5.4. Springs, Rows, and Pulleys

The force exerted by a spring is proportional to the distance the spring is stretched or compressed from its relaxed position

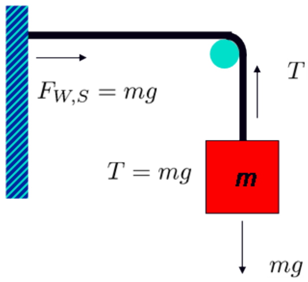
Hooke's Law

Hooke's Law

$$F_x = -kx$$

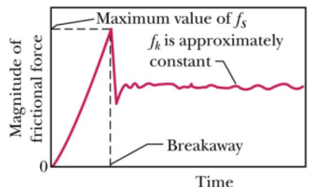
The proportional constant is called spring constant k and x denotes the displacement from rest position.

The tension T at a certain position in a rope is the magnitude of the force acting across a cross-section of the rope at that position. An ideal massless pulley or ideal smooth peg changes the direction of an applied force without altering the magnitude.



The figure depicts a rope mounted on a peg. The pulley redirects the rope from horizontal to vertical direction. The free body diagram of the block with mass m yields T = mg.

5.5. Friction



The figure illustrates the following experiment: a block of mass m rests on a table. You apply a force parallel to the table in order to shift the block. The block is not moving. You increase the force. Suddenly, the block brakes away. Once moving, the force you need to keep the speed constant is lower than the break away force.

The direction of the frictional force vector F is perpendicular to the normal force vector F_N in the direction opposing the relative motion of the two surfaces. Note that the static frictional force adapts to the force that you apply to the bock; it is not constant but has a maximum value. We denote μ_s and μ_k for the coefficient of static and kinetic friction, respectively.

Frictional Force (static)

$$f_{s,\max} = \mu_s F_N$$

Frictional Force (kinetic)

$$f_k = \mu_k F_N$$

5.6. Drag Forces

Anything that flows such as gas or liquid

Fluid

When fluid moves past a body, a force is applied on that body pointing into the direction of the relative flow.

Drag Force

$$D = \frac{1}{2} C \rho A v^2$$

C denotes the drag coefficient; and A, ρ, and v denote the effective cross-sectional area, the fluid density (mass per volume), and the constant velocity, respectively.

Since the drag force is proportional to the velocity squared, equilibrium is obtained in any free fall experiment (F_g = gravitational force).

Terminal Speed

$$v_t = \sqrt{\frac{2F_g}{C \rho A}}$$

6. Energy, Work and Power

Energy is one of the most important concepts in physics, since it provides an alternative approach to mechanics. In general, energy cannot be generated or destroyed, but it can be transformed or transferred.

6.1. Kinetic Energy

Kinetic energy E_K = K is energy associated with the state of motion of an object. The faster the object moves, the greater is its kinetic energy. When the object is stationary, its kinetic energy is zero.

Kinetic Energy

$$E_k = K = \frac{1}{2}mv^2$$

6.2. Work

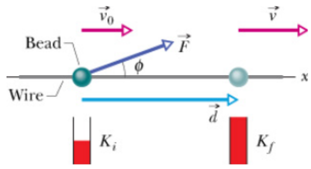
Energy transferred to or from an object by means of a force acting on the object

Work

Work

$$W = \vec{F} \cdot \vec{d}$$

where d is the displacement vector of the object. Energy transferred to the object is positive work W, and energy transferred from the object is negative work.



The figure illustrates a constant force F directed at angle ϕ to the displacement d of a bead on a wire accelerates the bead along the wire, changing its velocity from v_0 to v. The gauge indicates the initial and final kinetic energy, K_i and K_f, respectively.

Work (constant force)

$$W = Fd\cos\phi$$

Any change in kinetic energy done on a particle equals the net work done on the particle.

Kinetic Energy Theorem

$$W = \Delta E_k = E_{k2} - E_{k1} = \frac{1}{2}m(v_2^2 - v_1^2)$$

From the definition of work and the Kinetic Energy Theorem, we can calculate the work done by gravitational force on an object falling from height H.

Work (gravitational force, falling)

$$W_g = \vec{F} \cdot \vec{d} = mgH\cos(0) = mgH$$

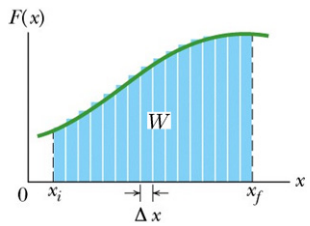
Speed (gravitational force, falling)

$$v = \sqrt{2gH}$$

Work (spring force)

$$W_s = \int_{x_1}^{x_2} -kx\ dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

The work W_s is positive if the object ends up closer to the relaxed position (x = 0) than it was initially. It is negative if object ends up farther away from x = 0. It is zero if the object ends up at the same distance from x = 0.



The figure depicts a variable force F(x) that changes in magnitude over the position x. The work done by that force is approximated by a constant force in small intervals Δx.

Work (variable force, one dimension)

$$W = \int_{x_1}^{x_2} F(x)\ dx$$

Work (variable force, three dimensions)

$$W = \int_{x_1}^{x_2} F(x)\ dx + \int_{y_1}^{y_2} F(y)\ dy + \int_{z_1}^{z_2} F(z)\ dz$$

6.3. Power

Time rate at which work is done *Power*

Power (average)

$$P_{\text{avg}} = \frac{W}{\Delta t}$$

Power (instantaneous)

$$P = \frac{dW}{dt}$$

If the work is done by a force, we obtain a vector equation:

Power (done by force)

$$P = \frac{dW}{dt} = \frac{F \cos \phi dx}{dt} = \vec{F} \cdot \vec{v}$$

7. Potential Energy and Conservation of Energy

7.1. Potential Energy

There is another type of energy, called potential energy $U = E_P$ that is associated with the configuration of a system of objects that exert forces on one another. Some examples are:

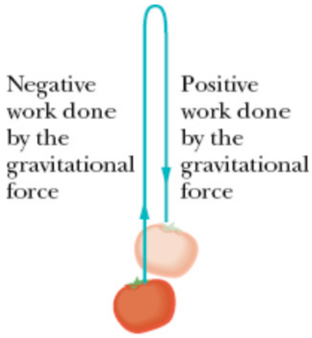
State of separation between objects, attracted by a gravitational force — *Gravitational Pot. Energy*

State of compression and extension of an elastic (spring-like) object — *Elastic Pot. Energy*

Potential Energy

$$\Delta U = -W$$

The figure depicts an apple that is thrown upwards into the air. Energy is transferred by the gravitational force to the gravitational potential energy of the object-Earth system. The work done is negative until the object stops. When the object falls down, the work W_g done on the object by the gravitational force is positive.



We consider a system of two or more objects. The configuration is changed by force F_1 that acts between particle-like object and rest of system. It transfers energy between object and rest of system. If the change is reversed by another force F_2, the energy transfer is also reversed.

The net work done by a force on a particle moving around any closed path is zero — *Conservative Force*

Work (conservative force)

$$W_1 = -W_2$$

Examples for conservative forces are the gravitational force and the spring force. Note, however, that kinetic frictional force or drag forces are non-conservative, since they transfer energy to another type (thermal energy).

When a force does work W on a particle-like object, the change in the potential energy associated with the system ΔU is the negative of the work done.

Potential Energy (system of particles)

$$\Delta U = -\int_{x_i}^{x_f} F(x)dx$$

Potential Energy (gravitational)

$$U(y) = mgy$$

Potential Energy (elastic)

$$U(x) = \frac{1}{2}kx^2$$

7.2. Mechanical Energy

Sum of potential and kinetic energies of all objects in the system — *Mechanical Energy*

Mechanical Energy

$$E_{\mathrm{mec}} = K + U$$

No external forces outside cause energy change inside the system — *Isolated System*

In an isolated system, only conservative forces can cause energy changes, where kinetic energy is transferred to potential energy, but the total mechanical energy E_{mec} of the system cannot change.

7.3. Conservation of Mechanical Energy

Mechanical Energy (isolated system)

$$\Delta E_{\mathrm{mec}} = \Delta K + \Delta U = 0$$

Energy cannot be generated or destroyed. Hence, we can relate the sum of kinetic and potential energies at two instances without considering any intermediate motion nor determining the work done by forces. This is much easier than applying Newton‘s laws.

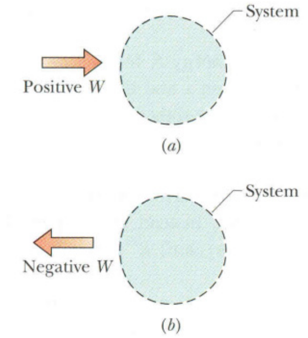
Work (general system)

$$W = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}}$$

Work (isolated system)

$$W = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}} = 0$$

The figure illustrates the definition of the sign of work done to a system. Positive and negative work is done if energy is transferred to or from a system, respectively.



8. Center of Mass and Linear Momentum

Analyzing complex motion of any sort requires simplification via an understanding of physics. If a complex structure is represented by it's center of mass, we can apply Newton's Laws to that point.

8.1. Center of Mass

A point that acts as (1) all of the system's mass is concentrated there and (2) all external forces are applied there — *Center of Mass*

Let i denote the index of a particle, m_i it's mass, and $r_i = (x_i, y_i, z_i)$ the position vector, we yield:

Center of Mass (system of particles)

$$\vec{r}_{\mathrm{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i \quad \Rightarrow \quad x_{\mathrm{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

In a solid body of mass M and volume V, we obtain:

Center of Mass (solid body)

$$x_{\mathrm{com}} = \frac{1}{M}\int x\,dm \quad , \quad x_{\mathrm{com}} = \frac{1}{V}\int x\,dV$$

Density

$$\rho = \frac{dm}{dV} = \frac{M}{V}$$

8.2. Linear Momentum

The term "momentum" may have several meanings in everyday language but only a single precise physical definition. The linear momentum is a vector quantity defined as mass times velocity.

Linear Momentum

$$\vec{p} = m\vec{v}$$

We can express Newton's 2^{nd} Law in terms of momentum: The rate of change of momentum equals the net force acting on the particle and is in direction of that force.

Newton's 2^{nd} Law (momentum)

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$$

Suppose that the net external force acting on a system of particles is zero (F = 0) and that no particles leave or enter the system, we find that a momentum can neither be created nor destroyed ("conserved") and the total momentum does not change with time. However, a momentum can be transferred.

Conservation of Momentum

$$\vec{p}_{(F=0)} = \mathrm{const}$$

8.3. Collision and Impulse

An isolated event in which two or more bodies exert relatively strong forces onto each other within a relatively short period of time — *Collision*

Newton's second law can be written as dp = F(t) dt. Let us integrate over the interval Δt that is from an initial time t_i to a final time t_f just before and after a collision of two bodies, respectively. We obtain:

Impulse (linear)

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p}$$

The right side of this equation is the change in linear momentum. The left side is a measure of both the strength and the duration of the collision force, is called the impulse J of the collision.

Momentum – Impulse Theorem (linear)

$$\Delta\vec{p} = \vec{J}$$

8.4. Inelastic Collision in 1D

Some energy of a system of two colliding bodies is transferred from kinetic energy to other forms of energy (e.g., thermal energy, sound) — *Inelastic Collision*

In an inelastic collision from initial (i) to final state (f) of two bodies m_1 and m_2 with velocities of v_1 and v_2, respectively, the momentum is conserved.

Momentum Conserved (general)

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

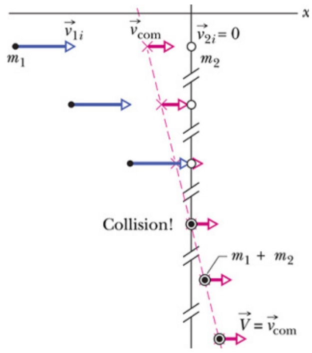
The two colliding bodies stick together — *Completely Inelastic Collision*

The inversion of a completely inelastic collision — *Explosion*

Velocity (completely inelastic collision)

$$v_f = \frac{m_1}{m_1 + m_2} v_{1i}$$

The figure visualizes that in a closed, isolated system, the velocity v_{com} of the center of mass of the system cannot be changed by a collision because, with the system isolated, there is no net extemal force to change it.



8.5. Elastic Collision in 1D

The total kinetic energy of a system of two colliding bodies is unchanged by the collision — *Elastic Collision*

Conservation of energy in an elastic collision means that the kinetic energy of each colliding body may change, but the total energy of the system does not change. Furthermore, the momentum is conserved.

Energy Conserved (general)

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

Elastic Collision (stationary object, v_{2i} = 0)

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \quad , \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Stationary Object (equal mass, $m_1 = m_2$)

$$v_{1f} = 0 \quad , \quad v_{2f} = v_{1i}$$

Stationary Object (massive target, $m_1 \ll m_2$)

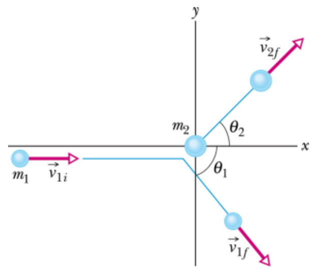
$$v_{1f} \approx -v_{1i} \quad , \quad v_{2f} \approx 2\frac{m_1}{m_2} v_{1i}$$

Stationary Object (massive projectile, $m_1 \gg m_2$)

$$v_{1f} \approx v_{1i} \quad , \quad v_{2f} \approx 2v_{1i}$$

8.6. Elastic Collision in 2D

The figure depicts an elastic collision between two bodies in witch the collision is not head-on (glancing collision). The target body (m_2) is initially at rest.



For such two-dimensional (2D) collisions in a closed, isolated system, the total linear momentum must still be conserved, and the total kinetic energy is also conserved.

Momentum Conserved

$$\vec{P}_{1i} = \vec{P}_{1f} + \vec{P}_{2f}$$

Energy Conserved

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

To solve the equations, the components of the vectors shall be separated.

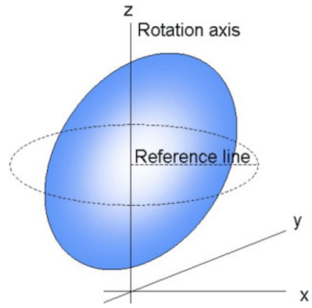
Elastic 2D Collision (x-component)

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$$

Elastic 2D Collision (y-component)

$$0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$$

9. Rotational Motion



The figure depicts a rotation of a rigid body (no change of shape) about a fixed axis of rotation. The position of the reference line with respect to the rigid body is arbitrary, but perpendicular to the rotational axis.

9.1. Rotational Variables

Every point in a body moves through the same angle during a particular time interval (pure rotation) — *Angular Motion*

Every point in a body moves the same distance on a straight line (pure translation) — *Linear Motion*

The rotational variables are defined analogously to the linear motion. The angular position of the reference line is the angle of the line relative to a fixed direction, which we take as the zero angular position.

Angular Position (radian measure)

$$\theta = \frac{s}{r}$$

Where s is the length of arc along a circle between the x axis (the zero angular position) and the reference line; r is the radius of that circle; and θ(t) is the angular position of the body's reference line as a function. Its unit is radian (rad).

$$1\ revolution = 360° = \frac{2\pi r}{r} = 2\pi\ rad$$
$$1\ rad = 57.3° = 0.159\ revolutions$$

The figure lists conversions from radian to revolutions and degree.

Angular Displacement

$$\Delta\theta = \theta_2 - \theta_1$$

Angular Velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Its unit is radian per second (rad/s).

Angular Acceleration

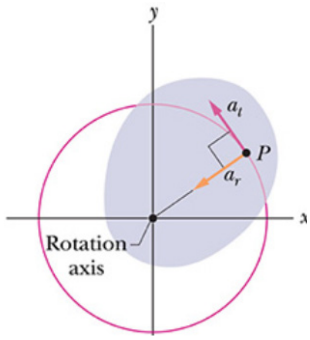
$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Its unit is radian per second-squared (rad/s²).

9.2. Rotational Variables as Vector Quantities

The figure visualizes the “right-hand rule”, relating rotational variables with a vector. Accordingly, we can treat angular velocity and acceleration as vectors pointing along the axis of rotation, where the length of the vector corresponds to the amount of the angular variable.

Angular displacement is not a vector, since it does not obey the rules of vector addition. However, we use the “right-hand rule” to assign the mathematical positive direction, which is counter clockwise.



As the figure suggests, we can relate the linear variables s, v, and a for a particular point in a rotating body to the angular variables θ, ω, and α for that body. The two sets of variables are related by r, the perpendicular distance of the point from the rotation axis.

Position

$$s = \theta r$$

Speed

$$v = \omega r$$

Tangential Acceleration

$$a_t = \alpha r$$

Centripetal Acceleration

$$a_r = \frac{v^2}{r} = \omega^2 r$$

9.3. *Newton's Second Law for Rotation*

The kinetic energy of a rotating body is the sum of the kinetic energies of its small particles:

Kinetic Energy (rotation)

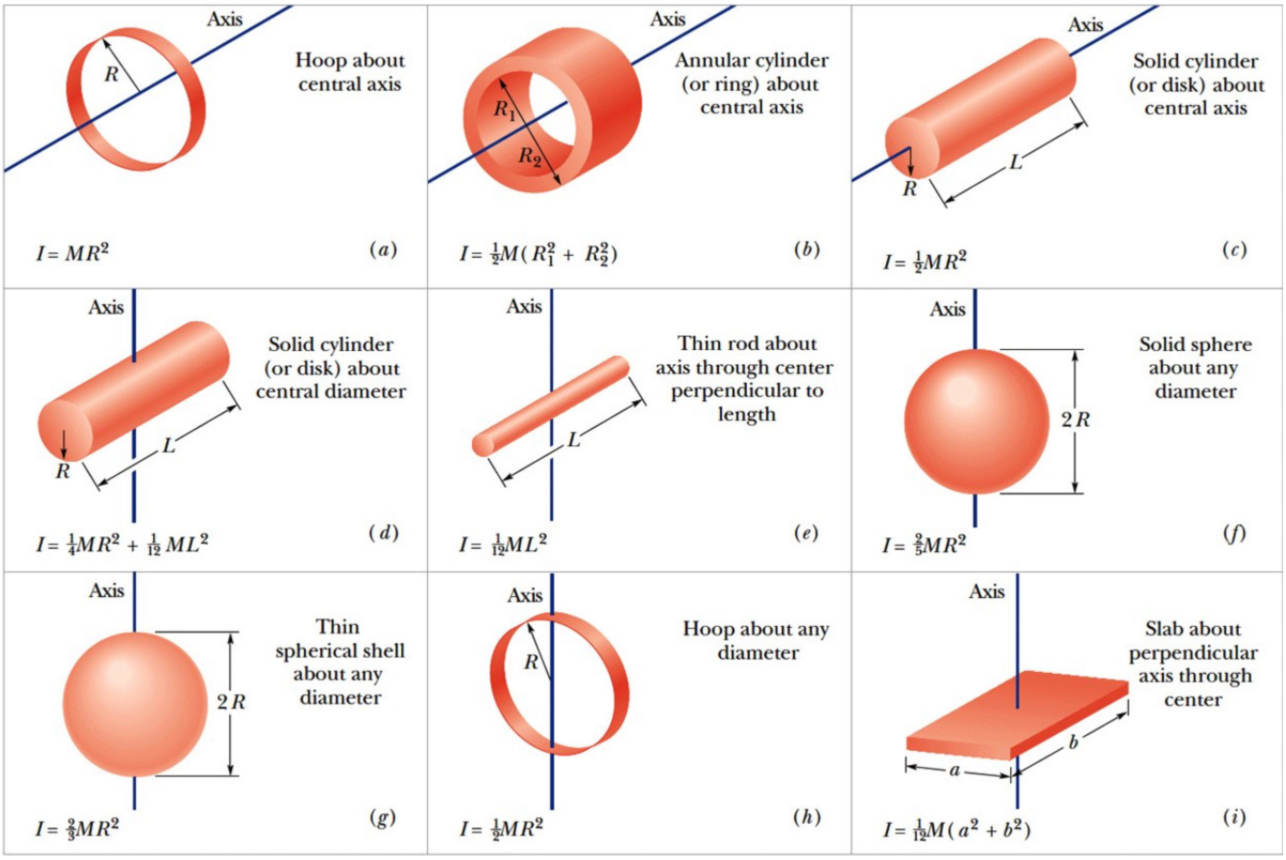
$$K = \frac{1}{2}mv^2 = \sum \frac{1}{2}m_i(\omega r_i)^2 = \frac{1}{2}I\omega^2$$

where I determines the moment of inertia (rotational inertia).

Moment of Intertia

$$I = \sum m_i r_i^2 \quad , \quad I = \int r^2 dm$$

The table lists some rotational inertias for different geometries.

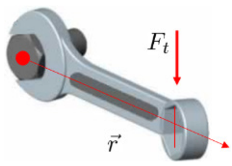


The rotational inertia I of a body of mass M about any axis parallel to an axis extending the body's center of mass (com) is calculated by the body's rotational inertia about that axis I_{com} and the distance d.

Parallel-Axis Theorem

$$I = I_{\text{com}} + Md^2$$

Inspired by a screw wrench (spanner, see figure), we find the ability of a force F to rotate a body depends not only on the magnitude of its tangential component F_t but also on how far from the origin the force is applied (moment arm r). We define a quantity called torque τ as the product of both.



Torque

$$\tau = rF\sin\phi = rF_t = r_\perp F$$

If the body would rotate counter clockwise, the torque is positive. If the object would rotate clockwise, the torque is negative. Torques obey the superposition principle: When several torques act on a body, the net torque (or resultant torque) is the sum of the individual torques.

Newton's Second Law (rotational)

$$\tau = I \cdot \alpha$$

9.4. Work by Rotational Motion

If a force F causes a rigid body of mass m to accelerate, it does work W on the body and thus the body's kinetic energy can change. The rotational situation is similar.

Work – Kinetic Energy Theorem

$$W = \Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

Work (rotation about fixed axis)

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

Work (constant torque)

$$W = \tau(\theta_f - \theta_i)$$

Power (rotation about fixed axis)

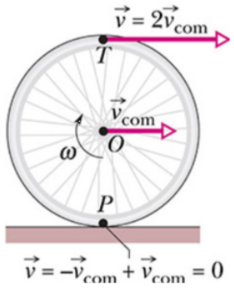
$$P = \frac{dW}{dt} = \tau\omega$$

9.5. Rotational and Linear Motion Compared

The following tables relate rotational and translational motions.

Pure Translation		Pure Rotation	
Position	X	Angular position	θ
Velocity	V	Angular velocity	ω
Acceleration	A	Angular acceleration	α
Mass	M	Rotational inertia	I
Newton's 2nd law	F_{NET}=ma	Newton's 2nd law	$\tau_{NET}=I\alpha$
Work	$W = \int F dx$	Work	$W = \int \tau d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power	P=Fv	Power	$P = \tau\omega$

10. Rolling, Angular Momentum and Equilibrium



10.1. Rolling

The figure depicts rolling as a complex movement combined from a linear motion of the center of mass with speed v_{com} and a rotational motion around that center with angular speed ω. The velocity of the top of the wheel is twice at that of its center O. The velocity at point P superimposes to zero.

Rolling Motion (smooth, no sliding)

$$v_{\text{com}} = \omega R$$

A rolling object has two types of kinetic energy: one due to the velocity of the center of mass (v_{com}), and another due to its rotation ω.

Kinetic Energy (rolling)

$$K = \frac{1}{2} I_{\text{com}} \omega^2 + \frac{1}{2} M v_{\text{com}}^2$$

An acceleration can make the rolling faster and slower. This acceleration also tends to make the wheel slide at contact point P. Thus, a frinctional force f_s must act on the wheel at P to oppose that tendency.

Acceleration (smooth rolling)

$$a_{\text{com}} = \alpha R \quad \Rightarrow \quad a_{\text{com,x}} = -\alpha R$$

The motion becomes more complex, if a round uniform body of mass M and radius R is rolling smoothly down a ramp of angle θ. The gravitational force F_g on the body is directed downwards. The normal force F_N is perpendicular to the ramp. The frictional force f_s acts on the point P and is directed upwards the ramp. This yields:

Acceleration (rolling down a ramp)

$$a_{\text{com,x}} = -\frac{g \sin\theta}{1 + I_{\text{com}}/MR^2}$$

10.2. Angular Momentum

Recall the concept of linear momentum and conservation of linear momentum as powerful tools. We now observe a particle with mass m add a linear momentum p = mv. That particle has the position r related to an origin. The angular momentum l is defined as

Angular Momentum (particle)

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Newton's Second Law (angular form, particle)

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt}$$

The total angular momentum L of a system of particles (e.g. a solid body) is the sum of the angular momentums of the individual particles.

Angular Momentum (body)

$$\vec{L} = \sum_{i=1}^{n} \vec{\ell}_i$$

The net external torque τ_{net} acting on a system of particles is equal to the time rate of change of the system's total angular momentum L.

Newton's Second Law (angular form, body)

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

If no net external torque acts on the system, we get the law of conservation of angular momentum.

Conservation of Angular Momentum

$$\vec{L} = \text{const}$$

If the component of the net external torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

10.3. Angular and Linear Momentum Compared

The following tables relate rotational and translational momentums.

Pure Translation		**Pure Rotation**	
Force	$\vec{F}$	Torque	$\vec{\tau} = \vec{r} \times \vec{F}$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{l} = \vec{r} \times \vec{p}$
Linear momentum (for systems of particles)	$\vec{P} = M\vec{v}_{COM}$	Angular momentum (for systems of particles)	$L = I\omega$
Newton's 2nd law (for systems of particles)	$\vec{F}_{NET} = \frac{d\vec{P}}{dt}$	Newton's 2nd law (for systems of particles)	$\vec{\tau}_{NET} = \frac{d\vec{L}}{dt}$
Conservation law (for an isolated system)	$\vec{P} = const.$	Conservation law (for an isolated system)	$\vec{L} = const.$

10.4. Equilibrium

We can find the requirements of static equilibrium, e.g., a system of masses that is neither in motion nor accelerated in linear or angular directions.

Balance of Forces

$$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt} = 0$$

The vector sum of all the external forces that act on the body in equilibrium must be zero.

Balance of Torques

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} = 0$$

The vector sum of all external torques that act on the body, measured about any possible point, must be zero.

With these findings, a hockey puck, for instance, is in equilibrium. For a static equilibrium, linear momentum and angular momentum must not only be constant, they also must be zero.

Equilibrium

$$\vec{P} = \text{const} \quad , \quad \vec{L} = \text{const}$$

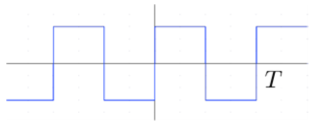
Static Equilibrium

$$\vec{P} = 0 \quad , \quad \vec{L} = 0$$

11. Oscillation

Motions that repeat themselves — *Oscillation*

The important property of an oscillatory motion is its frequency f, which determines the number of oscillations that are completed in each time interval. Its SI unit is hertz (1 hertz = 1 Hz = 1 oscillation per second = 1 s^{-1}).



As the figure depicts, the period T of the motion is the time that is required for one complete oscillation (cycle).

Period (Time Constant)

$$T = \frac{1}{f}$$

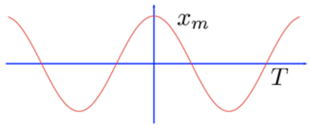
11.1. Harmonic Motion

Oscillation that repeats itself in a sinusoidal way — *Harmonic Motion*

The displacement x of the particle from the origin is given as a function of time by in which x_m, ω and ϕ are constants.

Displacement

$$x(t) = x_m \cos(\omega t + \phi)$$



As the figure depicts, the quantity x_m is called the amplitude of the motion. The subscript m stands for maximum because the amplitude is the magnitude of the maximum displacement of the particle in either direction.

The time-varying quantity (ωt + ϕ) is called the phase of the motion, and the constant ϕ is the phase constant (phase angle). The value of ϕ depends on the displacement and velocity of the particle at time t = 0. The constant ω is called the angular frequency of the motion.

Angular Frequency

$$\omega = 2\pi f$$

From the displacement, we can derive velocity and acceleration as first and second derivative over the time, respectively.

Velocity

$$v(t) = \frac{dx(t)}{dt} = -\omega x_m \sin(\omega t + \phi)$$

Acceleration

$$a(t) = \frac{dx^2(t)}{dt^2} = \frac{dv(t)}{dt} = -\omega^2 x_m \cos(\omega t + \phi)$$

Motion executed by a particle subject to a force that is proportional to the displacement but in opposite sign — *Linear Harmonic Motion*

A frictionless spring-mass system (spring constant k, mass m) performs such motion.

Period (linear motion)

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \Leftrightarrow \quad k = \omega^2 m$$

Projection of uniform circular motion on a diameter of the circle in which the circular motion occurs — *Angular Harmonic Motion*

A frictionless torsion pendulum, i.e., a rotating disk (rotational inertia I) mounted at a suspension wire (torsion constant κ) is an example.

Period (angular motion)

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad \Leftrightarrow \quad \kappa = \omega^2 I$$

11.2. Energy in Harmonic Motion

We now consider the energy of a linear harmonic oscillator. The potential energy of a spring-mass system is associated entirely with the spring (spring constant k), while the kinetic energy is associated entirely with the block (mass m).

Potential Energy

$$U = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$$

Kinetic Energy

$$K = \frac{1}{2}mv^2(t) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

The energy of a harmonic oscillator transfers back and forth between kinetic energy and potential energy, while the sum (total energy E) remains constant over time.

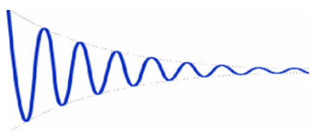
Total Energy

$$E = U + K = \frac{1}{2}kx_m^2 = \frac{1}{2}mv_m^2$$

11.3. Damped Harmonic Motion

Motion of an oscillator that is reduced by an external force *Damped Oscillator*

A pendulum will swing only briefly under water, because the water exerts a drag force on the pendulum that quickly eliminates the motion. A pendulum swinging in air does better, but still the motion dies out, which is not the case in vacuum.



The figure depicts a damping force F_d that is proportional in magnitude to the velocity v of the oscillator. The proportional constant b is called damping constant.

Displacement

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$$

Angular Frequency

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Mechanical Energy

$$E \approx \frac{1}{2}kx_m^2 e^{-\frac{bt}{m}}$$

11.4. Forced Oscillation

Oscillator that is pushed periodically *Forced Oscillator*

Two angular frequencies are associated with a system undergoing forced oscillations: the natural angular frequency ω of the system which and the angular frequency ω_d of the external driving force. A forced oscillator oscillates at ω_d.

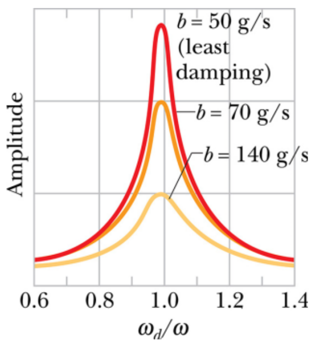
Displacement

$$x(t) = x_m \cos(\omega_d t + \phi)$$

Resonance

$$\omega_d = \omega$$

The amplitude x_m of the displacement is depending on a complicated function of ω_d and ω. The velocity amplitude v_m is greatest for resonance, which yields only approximately for the displacement.



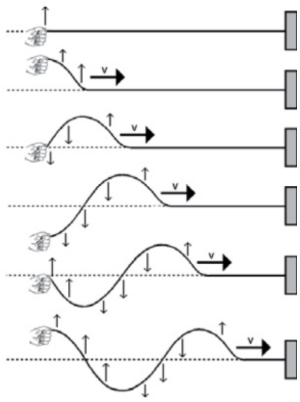
The figure displays resonance peaks for three different damping constants.

All mechanical structures have one or more natural angular frequencies (eigenfrequency), and if a structure is subjected to a strong external driving force that matches one of these eigenfrequencies, the resulting oscillations of the structure may rupture it. A prominent example is the Tacoma suspension bride.

12. Waves

Traveling disturbance that transports energy but not matter — *Wave*

There are different types of waves such as mechanical waves (seismic water, sound), where a matter is moved, electromagnetic waves (light), which do not require any medium, and matter waves (electrons, protons), which are associated to fundamental particles.



We also differ between transverse and longitudinal waves, where “particles” move perpendicular to (water molecules of water waves) and along with (air molecules of sound waves) the direction of travel of the wave, respectively.

The figure visualizes at a string that is periodically dislocated at one end. The displacement y of the element located at position x is dependent on the time t,

Displacement

$$y(x,t) = y_m \sin(kx - \omega t + \phi)$$

where y_m denotes the maximal amplitude, k the angular wave number, ω the angular frequency, and ϕ the phase constant. The phase $(kx - \omega t + \phi)$ changes linearly with time t.

The wavelength λ is the distance – parallel to the spreading direction of the wave – between repetitions of the wave shape at a fixed time t_0. The speed v of a wave is the covered distance divided by the time,

Speed

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

Where T is called time constant (period).

12.1. Wave Power and Energy

For wave propagation, each part of the medium communicates its motion to adjacent parts. Energy is transported by a wave. If we consider a wave along a string, kinetic energy refers to the movement of mass (string) and elastic potential energy to the tension τ of rope. The work is done by the hand, and a force is required. Both energies are maximal at y = 0, and they move with the motion of the wave,

Power

$$P(x,t) = \mu v \omega^2 y_m^2 \cos^2(kx - \omega t)$$

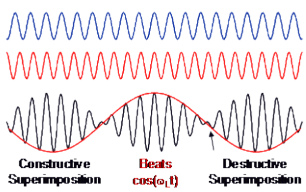
where μ denotes the string’s linear density (mass per unit length).

Speed (stretched string)

$$v = \sqrt{\frac{\tau}{\mu}}$$

12.2. Superimposition of Waves

Superimposition of waves means that they algebraically add, but the travel of each wave is not altered. They produce a resultant wave (net wave). We differ two cases: (i) constructively combination, where the amplitude is increased, and (ii) destructively combination, where the amplitude is decreased. Usually, both effects are present.



The figure depicts the superimposition of harmonic waves with frequencies $\omega_1 \neq \omega_2$.

Two waves superimpose with same wavelength, same amplitude, same direction, and different phase — *Interference*

Displacement

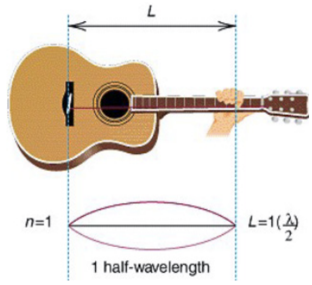
$$y'(x,t) = (2y_m \cos\frac{1}{2}\phi)\sin(kx - \omega t + \frac{1}{2}\phi)$$

Two superimposing waves with same amplitude, same wavelength, but in opposite direction — *Standing Wave*

Displacement

$$y'(x,t) = (2y_m \sin kx)\cos \omega t$$

At some places, called nodes, the string never moves. Halfway between adjacent nodes are antinodes where the amplitude is a maximum.



The figure depicts a guitar string that is stretched between two clamps separated by a fixed distance L. If the string starts oscillating, each of the clamps acts as a node. This restricts the oscillation to certain wavelengths, which is called resonance.

Resonance Frequency

$$f = \frac{v}{\lambda} = n\frac{v}{2L}$$

where n = 1, 2, … is an integer number (number of harmonics).

12.3. Wave Equation

In general, we can derive from Newton's 2nd Law a differential equation that governs the travel of waves of all types. It relates the second partial derivative of the displacement y to the direction x of travel with the second partial derivative to the time t, by a constant that is the inverted square of the wave speed v.

Wave Equation

$$\frac{\delta^2 y}{\delta x^2} = \frac{1}{v^2}\frac{\delta^2 y}{\delta t^2}$$

Anytime, the analysis of a physical system yields a general solution of this differential equation, the system supports wave propagation. An prominent example of such a man made system was the Tacoma suspension bridge.

General Solution

$$y = f(x - vt)$$

12.4. Sound Waves

Mechanical wave of pressure transmitted through a solid, liquid, or gas that is composed of frequencies within the range of hearing — *Sound*

The periodical movement of particles in gases, fluid, or solid state bodies of density ρ causes a periodical change in local pressure.

Displacement

$$s(x,t) = s_m \cos(kx - \omega t)$$

Pressure

$$\Delta p(x,t) = \Delta p_m \sin(kx - \omega t)$$

Pressure Amplitude

$$\Delta p_m = (v\rho\omega)s_m$$

The table lists some speeds of sound in different media, which is an inertial and an elastic property of the medium.

Medium	Speed [m/s]
Air (20°C)	343
Water (20°C)	1482
Steel	5941

The human ears percept frequencies from 20 Hz up to 20 kHz (the latter decreases with age). The threshold of hearing is $\Delta p_0 \approx 20$ μPa (Pascal, unit of pressure). The maximum pressure amplitude that human ears can tolerate is about $\Delta p_{max} = 28$ Pa. Hence, the human acoustic perception range exceeds six magnitudes.

Sound can cause glass to oscillate. If a standing wave with sufficient intensity is generated, the glass will shatter. Hence, sound waves transfer energy. The power of a sound wave determines the time rate of energy transfer.

Intensity

$$I = \frac{1}{2}\rho v \omega^2 s_m^2$$

Power

$$P_s = I \cdot A, \quad I = \frac{P_s}{4\pi r^2}$$

Sound Level

$$\beta = (10\text{ dB})\log\frac{I}{I_0} = (20\text{ dB})\log\frac{\Delta p}{\Delta p_0}$$

Where A denotes the area. The pseudo unit decibel (dB) is named after Alexander G. Bell.

12.5. Doppler Effect

If either the detector or the source (or both) is moving, the emitted frequency f and the detected frequency f' are related by an equation that is named after Johann Christian Doppler.

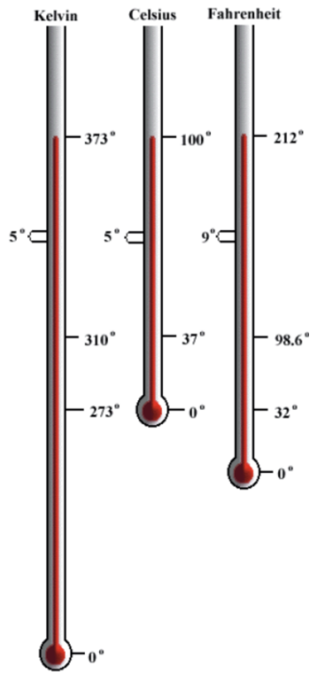
Doppler Equation

$$f' = \left(\frac{v \pm v_D}{v \mp v_S} \right) f$$

In this expression, the upper signs ($+v_D$ and $-v_S$) refer to motion of one toward the other, and the lower signs ($-v_D$ and $+v_S$) refer to motion of one away from each other; and "toward" and "away from" are associated with an increase and decrease in observed frequency, respectively.

13. Temperature and 1st Law of Thermodynamics

Study and application of thermal energy (internal energy) of systems — *Thermodynamics*



The core concept of thermodynamics is the temperature T, which is an SI base quantity related to our sense of hot and cold. It is measured with substances that change in a regular way their length or pressure, as they become hotter or colder.

The figure visualizes different scales of temperature.

Physicists measure temperature on the Kelvin (K) scale, which is the SI base unit. When the universe began, 13.7 billion years ago, its temperature was about 10^{39} K. Today, it has reached T ~ 3 K.

13.1. The 0th Law of Thermodynamics

The properties of many bodies change, as we alter their temperature. When two bodies have the same temperatures, they are in thermal equilibrium, and vice versa. More formal:

If bodies A and B are each in thermal equilibrium with a third body C (e.g., a thermoscope), then A and B are in thermal equilibrium with each other.

13.2. Measuring Temperature by Expansion of Matter

Liquid water, solid ice, and water vapour (gaseous water) can co-exist in thermal equilibrium at only one set of values for pressure and temperature. — *Triple Point of Water*

Triple Point Temperature (definition)
$$T_3 = 271.16\ \mathrm{K}$$

Celsius Scale
$$T_C = T - 273.15°$$

Fahrenheit Scale
$$T_F = \frac{9}{5}T_C + 32°, \quad T_C \approx \frac{(T_F - 30)}{2}$$

If the temperature of a metal rod is increased by an amount ΔT, its length L is found to increase by an amount ΔL. Accordingly, all dimensions of a solid with volume V expand.

Coefficient of Linear Expansion
$$\Delta L = L\alpha\Delta T, \quad \alpha = \frac{\Delta L}{\Delta T \cdot L}$$

Coefficient of Volume Expansion
$$\Delta V = V\beta\Delta T, \quad \beta = \frac{\Delta V}{\Delta T \cdot V} = 3\alpha$$

13.3. Heat

In mechanics, energy can be transferred between a system and its environment as work W via a force F acting on a system. However, there are other mechanisms of energy transfer, such as heat Q.

Energy that is transferred between a system and its environment because of a temperature difference — *Heat*

The SI unit of heat Q is joules (J), and 1 cal = 4.186 J is the amount of heat that would raise the temperature of 1 g water from 14.5°C to 15.5°C. Here, m denotes the mass of a mole of the substance.

Heat Capacity
$$C = \frac{Q}{\Delta T}, \quad Q = C\Delta T$$

Molar Specific Heat
$$c = \frac{C}{m}, \quad Q = cm\Delta T$$

The amount of energy required per unit mass to change the phase (but not the temperature) of a particular material

Heat of Transformation

Heat of Transformation

$$L = \frac{Q}{m}, \quad Q = Lm$$

Hence, we can define a heat of vaporization L_V (boiling or condensation) and a heat of fusion L_F (freezing or melting).

13.4. Work and Energy

A gas may exchange energy with its surroundings through work. The amount of work W done by a gas as it expands or contracts from an initial volume V_i to a final volume V_f must be integrated because the pressure p may change.

Work by Heat

$$W = \int dW = \int_{V_i}^{V_f} p \, dV$$

Both, heat Q and work W are path-dependent quantities. However, the principle of conservation of energy for a thermodynamic process yields the quantity Q – W is the same for all processes; the total energy is conserved.

1st Law of Thermodynamics

$$dE_{\text{int}} = dQ - dW$$

The internal energy E_{int} of a non-isolated system tends to increase if energy is added as heat Q, and it tends to decrease if energy is lost as work W done *by* the system. So far, we viewed only on work done *on* a system. Some special cases:

Adiabatic Process (Q = 0) $\qquad \Delta E_{int} = -W$

Constant-Volume Process (W = 0) $\qquad \Delta E_{\text{int}} = Q$

Cyclical Process (ΔE_{int} = 0) $\qquad Q = W$

Free Expansion (Q = W = 0) $\qquad \Delta E_{\text{int}} = 0$

Q represents the energy exchanged as heat between the system and its surroundings; Q is positive if the system absorbs heat, and negative if the system loses heat. W is the work done by the system; W is positive if the system expands against some external force exerted by the surroundings, and negative if the system contracts because of that external force.

13.5. Heat Transfer

Temperature differences between two systems can cause various types of energy transfer.

Energy transfer by friction of neighbored molecules

Conduction

We consider a slab whose faces are maintained at temperatures T_H and T_C by a hot and a cold reservoir, respectively. Heat is transferred by thermal conduction.

Conduction Rate

$$P_{\text{cond}} = \frac{Q}{t} = kA\frac{T_H - T_C}{L}$$

where the constant k is called the thermal conductivity; A and L refer to the face area and thickness of the slab, respectively.

Thermal Resistance

$$R = \frac{L}{k}$$

Energy transfer by motion within a fluid

Convection

Energy transfer via the emission of electromagnetic energy

Radiation

Radiation Rate

$$P_{\text{rad}} = \sigma \epsilon A T^4$$

Stefan-Boltzmann Constant

$$\sigma = 5.6703 \cdot 10^{-8} \, \frac{\text{W}}{\text{m}^2}\text{K}$$

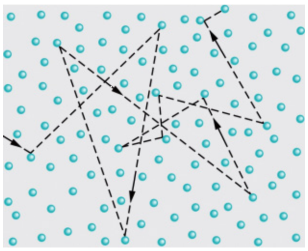
where A is the area of the object having an emissivity ε ($0 \leq \varepsilon \leq 1$), and T is the temperature of that area.

An object will radiate energy to its environment while absorbing energy from its environment with temperature T_{env}.

Absorption Rate

$$P_{\mathrm{abs}} = \sigma \epsilon A T_{\mathrm{env}}^4$$

14. Kinetics of Gases, Entropy, and 2nd Law



The motion of the atoms of a gas determines its volume V (resulting from the freedom to spread out), its pressure p (resulting from the collisions of the atoms), and its temperature T (due to kinetic energy of atoms).

The figure exemplifies a typical path of a molecule travelling through a gas. Speed and direction are changed abruptly on collisions, between the molecule moves on a straight line with constant speed. Note that although shown as stationary, the other molecules are also moving in a similar fashion.

A convenient way to specify the amount of a substance is the mole (mol), the number of atoms in a 12 g sample of carbon-12, where 1 mol = 6.02×10^{23} elementary units. The mole is one of the seven SI base units.

$N_A = 6.02 \times 10^{23}$ mol^{-1} *Avogadro's Number*

The number of moles n contained in a sample with mass M_s of any substance (molar mass M = mass of 1 mol) is equal to the ratio of the number of molecules N in the sample to the number of molecules N_A in 1 mol; where m denotes the mass of one molecule.

Number of Moles

$$n = \frac{N}{N_A} = \frac{M_s}{M} = \frac{M_s}{mN_A}$$

14.1. Ideal Gases

If we confine the same mole of samples of various gases in boxes of identical volume and same temperature, their measured pressures are nearly the same (not exactly); the small differences tend to disappear if the measurements are repeated at lower gas densities. At low enough densities, all gasses obey the following law.

Ideal Gas Law

$$pV = nRT = NkT$$

Gas Constant

$$R = 8.31 \frac{\mathrm{J}}{\mathrm{mol}} \cdot \mathrm{K}$$

Bolzmann Constant

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \frac{\mathrm{J}}{\mathrm{K}}$$

'Ideal' refers to the simplicity of the law that governs its macroscopic properties. Although in nature a truly ideal gas does not exist, all real gases approach the ideal state at low densities (molecules are far apart such that they do not interact).

The work W done by a gas depends on the type of process

Work (general process, ideal gas)

$$W = \int_{V_i}^{V_f} p \, dV = \int_{V_i}^{V_f} \frac{nRT}{V} \, dV$$

Work (constant temperature = isothermal)

$$W = nRT \int_{V_i}^{V_f} \frac{dV}{V} = nRT \ln \frac{V_f}{V_i}$$

Work (constant pressure)

$$W = p(V_f - V_i) = p\Delta V$$

Work (constant volume)

$$W = 0$$

14.2. Speed and Kinetic Energy

We now consider the movement of molecules of ideal gases. The root means square (RMS) speed can be derived from the pressure of a molecule exerted to the wall of the gas tank.

RMS Speed

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

Combining RMS speed and the ideal gas law is very much in the spirit of kinetic theory. It tells us how the pressure of the gas (a purely macroscopic quantity) depends on the speed of the molecules (a purely microscopic quantity).

Pressure (due to speed of molecules)

$$p = \frac{nMv_{\text{rms}}^2}{3V}$$

Now, we assume that the speed of the molecule changes when it collides with others.

Average Translational Kinetic Energy

$$K_{\text{avg}} = \frac{1}{2}m{v^2}_{\text{avg}} = \frac{1}{2}m\frac{3RT}{M} = \frac{3}{2}kT$$

At a given temperature T, all ideal gas molecules – no matter what their mass – have the same average translational kinetic energy K_{avg} = 3/2 kT. When we measure T of a gas, we also measure K_{avg} of its molecules.

Mean Free Path

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 N/V}$$

14.3. Entropy

An equivalent of the disorder of a system *Entropy*

If two isolated containers (A filled with an ideal gas; B is empty) are connected, the gas will irreversibly fill both containers evenly.

If an irreversible process occurs in a closed system, the entropy S of the system always increases, it never decreases *Entropy Postulate*

Because of this property, the change in entropy is called “the arrow of time”. Since a backward process would result in an entropy decrease, it never happens.

We already know the state properties pressure p, volume V, energy E and temperature T. We now define a new state property entropy S.

Change in Entropy (definition)

$$\Delta S = S_f - S_i = \int_i^f \frac{dQ}{T}$$

This change depends not only on the energy transferred as heat Q but also on the temperature T at which the transfer takes place. To find ΔS for an irreversible process occurring in a closed system, we replace it with a reversible isothermal process (state property).

Change in Entropy (isothermal process)

$$\Delta S = S_f - S_i = \frac{1}{T}\int_i^f dQ = \frac{Q}{T}$$

If an ideal gas changes reversibly from an initial state (T_i, V_i) to a final state (T_f, V_f), the change in the entropy ΔS is:

Change in Entropy (state function)

$$\Delta S = S_f - S_i = nR\ln\frac{V_f}{V_i} = nC_V\ln\frac{T_f}{T_i}$$

Molar Specific Heat at Constant Volume (mono-atomic gas)

$$C_V = \frac{3}{2}R = 12.5\,\frac{\text{J}}{\text{mol}}\text{K}$$

14.4. 2nd Law of Thermodynamics

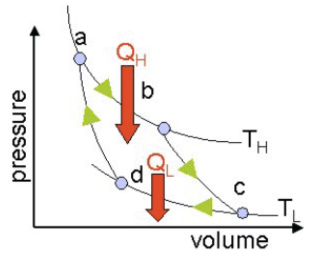
If a process occurs in a closed system, the entropy of the system either increases (irreversible process) or remains constant (reversible process). However, the entropy never decreases.

2nd Law of Thermodynamics

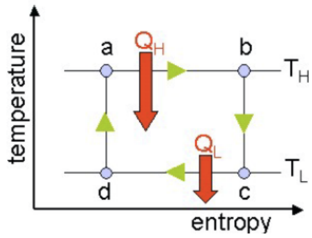
$$\Delta S \geq 0$$

14.5. Carnot Engine

In 1824 an ideal engine was proposed by Sadi Carnot, in which all processes are reversible and no energy is wasted. A Carnot engine has a working substance (eg. gas) in an insulating cylinder with a movable piston. The cylinder may be placed on a hot T_H or cold reservoir T_L or on an insulating plate.



The figure shows a Carnot engine's pV-diagram. The cycle consist of two isothermal (a→b and c→d) and two reversible adiabatic (b→c and d→a) processes. During the processes a→b (expansion) and c→d, the temperature remain constant; and during b→c and d→a, the entropy remains constant.



The figure shows a Carnot engine's TS-diagram. Heat Q is transferred in the isothermal processes. During a→b and b→c, the working substance is expanding. This raises the piston and thus is doing positive work (area under curve a→b→c). During c→d and d→a, the working substance is compressed doing negative work (area under curve c→d→a).

Work (Carnot engine)

$$W = |Q_H| - |Q_L|$$

Entropy Changes (Carnot engine)

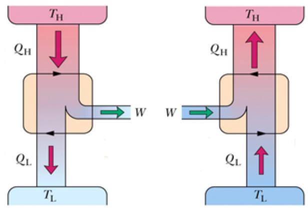
$$\Delta S = \frac{|Q_H|}{T_H} - \frac{|Q_L|}{T_L} = 0$$

Efficiency (Carnot engine)

$$\epsilon_C = \frac{|W|}{|Q_H|} = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{T_L}{T_H}$$

14.6. Ideal Refrigerator

With an ideal refrigerator, all processes are reversible and no energy is wasted. It can be derived from the Carnot machine, where work W is used to transfer energy Q_L from the low-temperature reservoir (inside the fridge) to the high-temperature reservoir Q_H (room outside).



The figure visualizes a Carnot engine (left) and the Carnot refrigerator (right).

Again, a coefficient of performance K relates "what we want" to "what we pay for". The value of K is higher the closer the temperatures of each reservoirs.

Coefficient of Performance (Carnot refrigerator)

$$K_C = \frac{|Q_L|}{|W|} = \frac{|Q_L|}{|Q_H| - |Q_L|} = \frac{|T_L|}{|T_H| - |T_L|}$$

15. Electric Charge & Fields and Gauss' Law

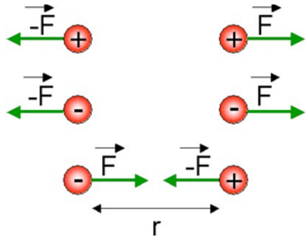
15.1. Electric Charge

An intrinsic property of particles, being either positive or negative — *Electric Charge*

Elementary Charge

$$e = 1.602 \times 10^{-19}\mathrm{C}$$

The figure depicts that like charges q attract and unlike repel. The electrostatic force F acts along the line r connecting the two charges.



If an object consists of several charges, F is calculated by superposition. The net force of each charge is calculated as the vector sum of the forces exerted on the charge by all the others. If an object consists of the same amount of negatively and positively charged particles, it is electrically neutral, i.e. it contains no net charge.

Electrostatic Force (Coulomb's Law)

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1| \cdot |q_2|}{r^2}$$

Permittivity Constant

$$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$$

Two charged particles with the same amount of charge q but different sign that are mounted with a small distance d — *Electric Dipole*

Dipole Moment (points from negative to positive)

$$\vec{p}, \quad p = qd$$

Material in which charged particles are free to move (metal) — *Conductor*

Material without freely movable charged particles (glass, plastic) — *Insulator*

Material intermediate between conductors and isolators (silicon) — *Semiconductor*

Material in which charge can move absolutely freely (no resistance) — *Superconductor*

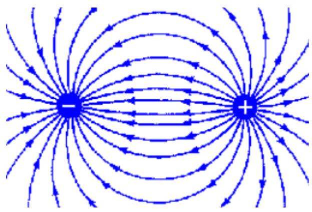
15.2. Electric Field

Electrostatic force that would be exerted on a positive test charge q_0 — *Electric Field*

Electric Field

$$\vec{E} = \frac{\vec{F}}{q_0}$$

This defines a vector field. Electric field lines visualise the direction and magnitude of the field. Every field vector is tangent to the field lines. The density of field lines in a region is proportional to the magnitude of the electric field in that region.



The figure depicts that positive field lines are defined to extend away from positive charge toward negative charge.

Electric Field of Point Charge (magnitude)

$$E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2}$$

Electric Field of Dipole (magnitude, distance z along dipole axis)

$$E = \frac{1}{2\pi\epsilon_0}\frac{p}{z^3}$$

The electric field inside a conductor is always zero (Faraday cage).

15.3. Gauss' Law

Hypothetical closed surface of any shape (select convenient) enclosing a net charge — *Gaussian Surface*

The mechanic flux was defined as the rate of flow through a surface. Accordingly, the electric flux is defined as field lines passing through a surface. The SI unit is the newton square-meter per coulomb ($\mathrm{Nm^2/C}$).

Electric Flux

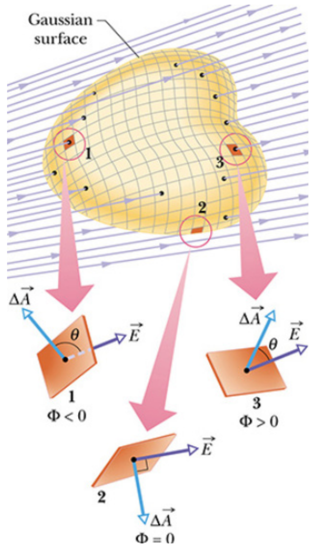
$$\Phi_E = \sum \vec{E} \cdot \Delta\vec{A}, \quad \Phi_E = \oint \vec{E} \cdot d\vec{A}$$

Gauss' Law relates the flux of an electric field through a closed surface (Gaussian surface) to the net charge q_{enc} that is enclosed by that surface. It is equivalent to Coulomb's Law; each can be derived from the other.

Gauss' Law

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}}, \quad \epsilon_0 \Phi = q_{\text{enc}}$$

If q_{enc} is positive, the net flux is outward, if q_{enc} is negative, the net flux is inward. Charges outside the Gaussian surface don't contribute to the integral because as many field lines enter the surface as leave it.



The figure depicts a Gaussian surface. Both, area A and electric field E are vector quantities, related by a dot product.

16. Electric Potential and Capacitance

16.1. Charges in an Electric Field

If any charge q is placed into an electrical field E, an electrostatic force F results, which can be derived from our definition of the electrical field.

Electrostatic Force on Charge $\vec{F} = q \cdot \vec{E}$

The motion of a charged particle in an electrical field equals that of a point mass in the gravitational field.

For a dipole of moment p, the net charge q_{net} is zero and hence the net force F_{net} is zero (uniform field). However, an electrostatic torque τ is acting.

Electrostatic Torque on Dipole $\vec{\tau} = \vec{p} \times \vec{E}, \quad \tau = -pE\sin\theta$

16.2. Electric Potential and Energy

Energy of a charged object in an external electric field (unit: joules) — *Electric Potential Energy*

If a system consists out of two or more charged particles, we can assign an electric potential energy U to the system. If the system changes from state i (initial) to state f (final), the electrostatic force does work W on the particles.

Electrostatic Work $\Delta U = U_f - U_i = -W$

This work is path-independent. For convenience we define the reference potential energy of a particle zero, if it is infinitely separated from the others: $U_\infty = 0$.

Potential energy per unit charge at a point in an electric field — *Electric Potential*

Electric Potential Difference $\Delta V = \frac{\Delta U}{q}$

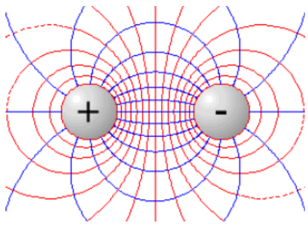
The electric potential is a property of an electric field; never mind if a charged object has been placed in that field. It is measured in Joules per Coulomb.

Electric Potential $V = -\frac{W_\infty}{q}$

where W_∞ is the work done by an electric field on a charged particle that is moved from infinity to the considered point.

Adjacent points having the same electric potential — *Equipotent Surface*

As the figure shows, equipotent surfaces (red) are always perpendicular to electric field lines (blue). No net work W is done on a charged particle by an electric field when the particle is moved on that surface.



16.3. Capacitance and Capacitor

A device to store electric energy by means of charge — *Capacitor*

An ideal capacitor consists of two isolated conductors (the plates) with equal charge in magnitude but opposite signs: +q and -q.

The electrical property of the capacitor — *Capacitance*

Capacitance $C = \frac{V}{q}, \quad q = C\Delta V = CV$

V is the potential difference between the plates. The SI unit of capacitance is the farad (F) or coulomb per volt (F = C/V). It depends on the geometry of the capacitor. If A denotes the area, L the length, and a and b the inner and outer radii, respectively, we obtain applying Gauss' Law:

Capacitance (parallel)
$$C = \frac{\epsilon_0 A}{d}$$

Capacitance (cylindric)
$$C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)}$$

Capacitance (spherical)
$$C = 4\pi\epsilon_0 \frac{ab}{b-a}$$

16.4. Energy Stored in a Capacitance

The electric potential energy U of a charged capacitor is equal to the work required to charge it. This energy can be associated with the capacitor's electric field E.

Potential Electric Energy in Capacitor
$$U = \frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$

The potential energy of a charged capacitor may be viewed as being stored in the electric field between its plates.

By extension, we can associate stored energy with an electric field. In vacuum, the energy density u (potential energy per unit volume) within an electric field of magnitude E is given by

Energy Density in Capacitor
$$u = \frac{1}{2}\epsilon_0 E^2$$

16.5. Capacitor Filled with Dielectric

If the space between the plates of a capacitor is completely filled with a dielectric material, the capacitance C is increased by a factor κ, which is called the dielectric constant. It is a characteristic of the material.

Material	Constant κ	Strength (kV/mm)
Vacuum	1.00000	-
Air	1.00054	3
Teflon	2.1	16
Polystyrene	2.6	24
Waxed paper	3.3 – 3.5	40 – 60
Water	80	-

The table lists some dielectric constants of different material.

Dielectric Constant
$$\kappa = \frac{C}{C_0}, \quad C = \kappa C_0$$

Faraday has observed that if a region is completely filled by a dielectric, all electrostatic equations containing ε_0 must be modified by replacing ε_0 with $\kappa\varepsilon_0$.

16.6. Capacitance of Combined Capacitors

If two capacitors C_1 and C_2 are combined in parallel, both ends connect together by a wire. Hence, the resulting equivalent Capacitor C_{eq} has the same voltage ($V_{eq} = V_1 = V_2$), the areas add ($C_{eq} = C_1 + C_2$), and the charge is shared ($q_{eq} = q_1 + q_2$).

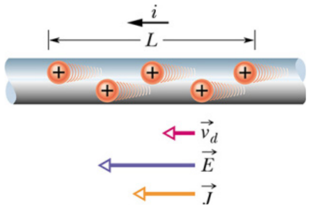
Capacitors in Parallel
$$C_{\mathrm{eq}} = \sum_{j=1}^{N} C_j$$

If two capacitors C_1 and C_2 are combined in series, the situation likes a capacitor partly filled with a conductor. Hence, they have the same charge ($q_{eq} = q_1 = q_2$) but share the voltage ($V_{eq} = V_1 + V_2$). The distances add ($d_{eq} = d_1 + d_2$), where d ~ 1/C.

Capacitors in Series
$$\frac{1}{C_{\mathrm{eq}}} = \sum_{j=1}^{N} \frac{1}{C_j}$$

17. Current and Resistance

The figure visualizes a moving charge in a conductor.



Electric Current

17.1. *Electric Current*

Stream of moving charge

An electric current i in a conductor is defined by the amount of (positive) charge dq that passes in time dt through a hypothetical cut across the conductor. By convention, the direction is taken as the direction in which positively charged carriers would move.

Electric Current (scalar)

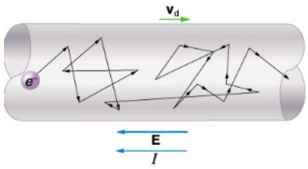
$$i = \frac{dq}{dt}$$

At a particular point of the conductor, the flow of charge J through that cross section of area A yields the

Current Density (vector)

$$J = \frac{i}{A}, \quad i = \int \vec{J} \cdot d\vec{A}$$

has the same direction as the velocity of the moving charges if they are positive and the opposite direction if they are negative.



The figure shows that electrons of elementary charge e in a wire move randomly. If the conductor has a current, they still move randomly but tend to have a drift with speed v_d.

Drift Speed

$$\vec{v}_d = \frac{\vec{J}}{(ne)}, \quad \vec{J} = (ne)\vec{v}_d$$

where n denotes the number of carriers per volume unit.

17.2. *Resistance and Resistivity*

We can determine the resistance between any two points of a conductor by applying a potential difference V and measuring the current i. The SI unit is the ohm (Ω) or volt per ampere (Ω = V/A).

Resistance

$$R = \frac{V}{i}$$

Specific resistance in an electrical circuit

Resistor

Similar equations define the resistivity ρ and conductivity σ, which both are properties of the material.

Resistivity

$$\rho = \frac{E}{J}, \quad \vec{E} = \rho\vec{J}$$

Conductivity

$$\sigma = \frac{1}{\rho}$$

Resistance (uniform wire of length L, cross section area A)

$$R = \rho\frac{L}{A}$$

The table gives the typical resistivity of conductors, semiconductors, and isolators and 20° C room temperature.

Material	Resistivity, ρ ($\Omega \cdot m$)
	Typical Metals
Silver	1.62×10^{-8}
Copper	1.69×10^{-8}
Gold	2.35×10^{-8}
Aluminum	2.75×10^{-8}
Manganin[a]	4.82×10^{-8}
Tungsten	5.25×10^{-8}
Iron	9.68×10^{-8}
Platinum	10.6×10^{-8}
	Typical Semiconductors
Silicon, pure	2.5×10^{3}
Silicon, *n*-type[b]	8.7×10^{-4}
Silicon, *p*-type[c]	2.8×10^{-3}
	Typical Insulators
Glass	$10^{10} - 10^{14}$
Fused quartz	$\sim 10^{16}$

17.3. *Ohm's Law*

We distinguish between two types of devices by saying the one obeys Ohm's law and the other does not.

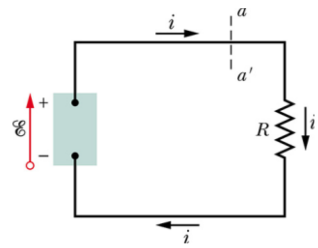
A conducting device obeys Ohm's law when the resistance of the device is independent of the magnitude and the polarity of the applied potential difference.

The current through a device is always directly proportional to the potential differences applied to that device

Ohm's Law

18. Electric Circuits

The figure depicts a simple circuit of battery and resistive device.



18.1. Power in Electric Circuits

We consider a closed loop with charged capacitor and resistor. A current is observed that lowers the potential energy of the capacitor. From energy conservation, we conclude that the decrease of electric potential energy equals the increase of transferred type of energy.

Power (rate of energy transfer)

$$P = iV$$

Power (resistive dissipation)

$$P = i^2R = \frac{V^2}{R}$$

18.2. Electromotive Force (emf)

When a capacitor discharges, the current decreases continuously. To establish a steady flow of charge, we need a "charge pump" (e.g., battery, electric generator, solar cell, fuel cell), a device $\mathcal{E}$ that – by doing work on the charge carriers – maintains the potential difference.

A device maintaining the potential differences between its terminals — *Electromotive Force (emf)*

An direct current (DC) emf device lacks any internal resistance.

DC emf

$$\mathcal{E} = \frac{dW}{dq}$$

Current (single-loop circuit)

$$i = \frac{\mathcal{E}}{R}$$

For a move through an ideal emf device in the direction of the emf arrow, the change in potential is $+\mathcal{E}$; in the opposite direction it is $-\mathcal{E}$ — *emf Rule*

18.3. Kirkhoff's Laws

The sum over all potential differences falling along an electrically connected loop of a circuit is zero — *Loop Rule*

When resistances R are in series, they have the same current i. The equivalent resistance R_{eq} that replaces a series combination is

Resistors in Series

$$R_{\text{eq}} = \sum_j R_j$$

At one specific spot of an electrical circuit, the sum of all ingoing currents is equal to all outgoing currents — *Junction Rule*

When the resistances are in parallel, they have the same potential difference falling between their ends. The equivalent resistance R_{eq} is

Resistors in Parallel

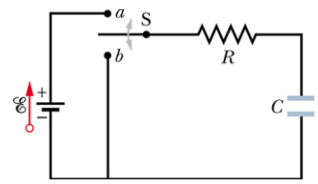
$$\frac{1}{R_{\text{eq}}} = \sum_j \frac{1}{R_j}$$

Resistors in Parallel (2 resistors)

$$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}$$

18.4. RC Circuits

The figure shows a simple RC circuit, where the switch is used to charge or discharge the capacitor.



To describe the circuit, we apply Kirkhoff's loop rule. After the switch connects the battery, the potential difference over the resistor and the capacitor equals that over the emf.

Kirkhoff's Loop

$$\mathcal{E} - iR - \frac{q}{C} = 0, \quad i = \frac{dq}{dt}$$

Charge (charging a capacitor)

$$q = C\mathcal{E}(1 - e^{-t/RC})$$

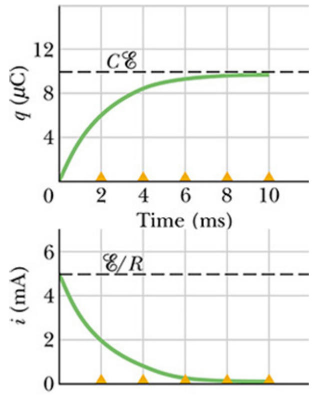
Current (charging a capacitor)

$$i = \frac{dq}{dt} = \left(\frac{\mathcal{E}}{R}\right) e^{-t/RC}$$

The plots visualize the build-up of charge q in a capacitor and the according current i. The curves are plotted for R = 2000 Ω, C = 1 μF, and $\mathcal{E}$ = 10 V. Their exponential shape is clearly observed.

A capacitor that is charged initially acts like an ordinary connecting wire relative to the charging current. A long time later, it acts like a broken wire.

The product RC has the dimension of time. It is called the capacitive time constant.



Time Constant (capacitive)

$$\tau = RC$$

When a capacitor discharges through a resistance R, the charge on the capacitor decays accordingly.

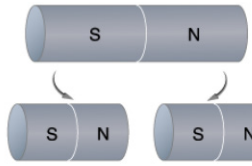
Charge (discharging a capacitor)

$$q = q_0 e^{-t/\tau}$$

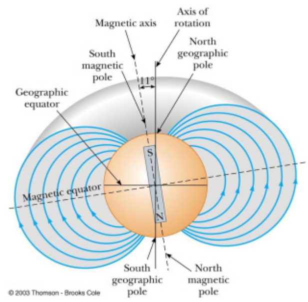
Current (discharging a capacitor)

$$i = -\left(\frac{q_0}{RC}\right) e^{-t/\tau}$$

19. Magnetic Fields



The figure indicates that a permanent magnet has a North pole and a South pole. We can observe magnetic forces: unlike poles attract and likes poles repel. We further observe the inseparability of North and South poles, which are always occurring in pairs.



The figure depicts the magnetic field of the Earth. The North magnetic pole is close to South geographic pole.

A magnetic field can be produced from permanent magnets (such as the Earth) or from an electromagnet.

19.1. Lorentz Force

As the Coulomb force acts on a charge in an electrical field E in parallel to the field lines, the Lorentz force acts on a *moving* a charge in a magnetic field B. It is directed always perpendicular to both, magnetic field and velocity (right hand rule).

Magnetic Force (vector)

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

The angle ϕ is defined between the magnetic field and the direction of the moving charge.

Magnetic Force (magnitude)

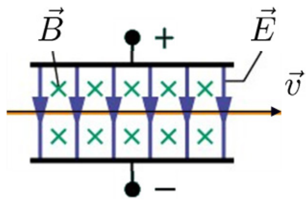
$$F_B = |q| v B \sin\phi$$

Magnatic Field (magnitude)

$$B = \frac{F_B}{|q| v \sin\phi}$$

The SI unit for B is the tesla (T) or newton per ampere-meter (T = N/Am) or formerly, the gauss (G = 10^{-4} T). Unlike in electricity, there is no magnetic charge. A magnetic field is always composed of closed lines; there are neither sources nor sinks.

19.2. Magnetic and Electric Field



The figure shows a constant magnetic field B superimposed to a charged capacitor with a constant electric field E between the plates.

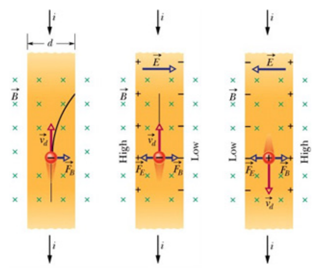
We can determine magnitude and direction of B such that a positively charged particle with initial velocity v travels straight through the capacitor. The equilibrium of forces yields (E = F/q)

Magnetic Field (magnitude)

$$B = \frac{E}{v}$$

This is the core setting of any cathode ray tube, invented by Sir John Josef Thomson (1895).

19.3. Hall Effect



The figure emphasise that when a moving charge is immersed in a magnetic field B, an electric field E is induced. This field is telling us the sign of the charges and number n of charged particles.

The Hall potential difference V = Ed in a of strip of width d thickness l = A/d then yields

Hall Equation

$$n = \frac{Bi}{Vle}$$

A charged particle with mass m and charge q moving with velocity v perpendicular to a uniform magnetic field B will travel in a circle of radius r. Its circular path is described by

Period

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{|q| B}$$

Frequency

$$f = \frac{1}{T} = \frac{|q|B}{2\pi m}$$

Angular Frequency

$$\omega = 2\pi f = \frac{|q|B}{m}$$

Radius

$$r = \frac{mv}{|q|B}$$

Using the Hall equation, we can determine the magnetic force on a current-carrying wire of length L directed along the current.

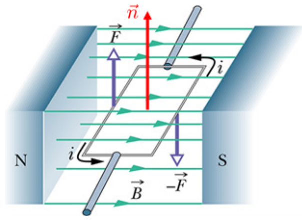
Magnetic Force (straight wire)

$$\vec{F}_B = i\vec{L} \times \vec{B}$$

Magnetic Force (curved wire)

$$d\vec{F}_B = i\ d\vec{L} \times \vec{B}$$

19.4. Magnetic Torque and Dipole Moment



The figure depicts a setting to construct a motor. The looped wire is carrying a current. The magnetic forces superimpose to a magnetic torque that tries to align n with B.

The loop's area is A = 2ab, where the length a and b denote the wire segments perpendicular and along the magnetic field, respectively.

Magnetic Torque (magnitude)

$$\tau = (iab)B\sin\phi$$

Magnetic Torque (vector)

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Obviously, a coil of area A and N loops behaves as magnetic dipole μ. Hence, we define the

Magnetic Dipole Moment

$$\mu = NiA$$

It's SI unit is ampere times square-meter (Am^2) or Jules per Tesla (J/T).

The table lists some magnetic dipole moments.

Some Magnetic Dipole Moments

Small bar magnet	5 J/T
Earth	8.0×10^{22} J/T
Proton	1.4×10^{-26} J/T
Electron	9.3×10^{-24} J/T

According to electric dipoles in the electric field, we derive

Magnetic Potential Energy

$$U = -\vec{\tau} \cdot \vec{B}$$

If the system changes from state i (initial) to state f final, the magnetic force does work on the magnetic dipoles

Work (on a stationary dipole)

$$W = U_f - U_i$$

19.5. Biot-Savart's Law

The figure illustrates that a current-carrying wire induces a magnetic field B. Iron files are used to visualize B.

The B that is set up by a current-carrying conductor can be found from the contribution dB to the field produced by a current-length element i ds in a distance r from the current element.

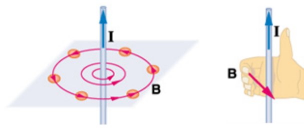
Biot-Savart's Law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\ d\vec{s} \times \vec{r}}{r^2}$$

Permeability Constant

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$$

Using Biot-Savart's law, we can determine the magnitude of magnetic fields in special wire geometries in a distance R from the wire.



The figure depicts the right hand rule to determine the orientation of the magnetic field B due to a wire carrying a current i.

Magnetic Field (long straight wire)

$$B = \frac{\mu_0 i}{2\pi R}$$

Magnetic Field (semi-infinite stright wire)

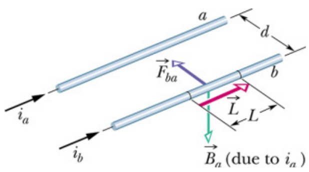
$$B = \frac{\mu_0 i}{4\pi R}$$

Magnetic Field (center of circular arc of angle ϕ)

$$B = \frac{\mu_0 i\phi}{4\pi R}$$

Magnetic Field (center of closed circle)

$$B = \frac{\mu_0 i}{2R}$$

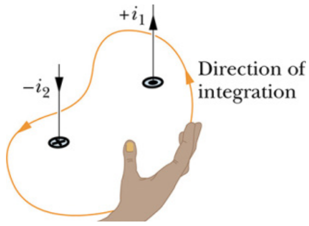


The figure sketches parallel wires. Combining Hall effect and Biot-Savart's law determines the electromagnetic force between the wires, where F_{ba} denotes the force on wire b due to the magnetic field of wire a, and L denotes the length of the wires and d their distance.

Force (between wires)

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}$$

19.6. Ampere's Law



The figure shows the principle of an Ampere's loop, a similar concept that we have already been used: the Gaussian surface.

The Ampere's Law states, that the line integral of the magnetic field evaluated around a closed loop – called an Amperian loop – is proportional to the net current in the loop. The current i is the net current encircled by the loop.

Ampere's Law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

For some current distributions, Ampere's law is easier than the Biot-Savart's law to calculate the magnetic field due to the currents.

Magnetic Field (inside a wire)

$$B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r$$

Magnetic Field (solenoid)

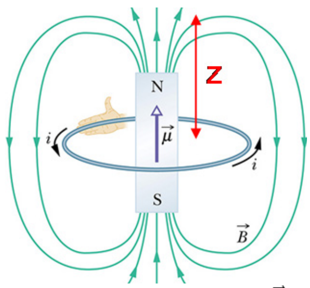
$$B = \mu_0 i N$$

Magnetic Field (toroid)

$$B = \frac{\mu_0 i N}{2\pi} \frac{1}{r}$$

Magnetic Field (coil, magnitude)

$$B(z) = \frac{\mu_0}{2\pi} \frac{NiA}{z^3}$$



Here, N denotes the number of loops in the according geometry, r < R the radius inside the wire or the toroid, z the distance from the coil along the coil z-axis, and A = π R^2 the cross sectional area of the coil.

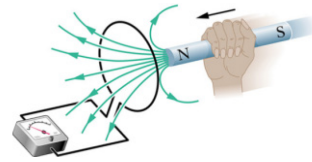
From the figure, we can regard a current-carrying coil as a magnetic dipole: (i) it experiences a torque when be placed in an external magnetic field and (ii) generates it own magnetic field.

Magnetic Field(coil, vector)

$$\vec{B}(z) = \frac{\mu_0}{2\pi} \frac{\vec{\tau}}{z^3}$$

20. Induction and Inductivity

20.1. *Magnetic Flux and Faraday's Law*



The figure demonstrates that a current is induced in a wire loop when the number of magnetic field lines passing through the loop is changing.

Magnetic Flux

$$\Phi_B = \oint \vec{B} \cdot d\vec{A}$$

We define the magnetic flux Φ_B through an area A in a magnetic field B. Its SI unit is the weber (W) or tesla times square-meter (W = Tm^2).

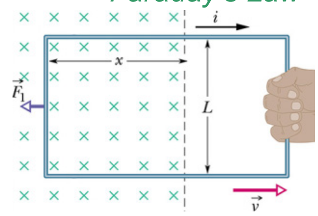
Production of a potential difference (emf) across a conducting loop due to a varying magnetic field — *Induction*

Faraday's Law

$$\mathcal{E} = -N\frac{d\Phi_b}{dt} = \oint \vec{E} \cdot d\vec{s}$$

The magnitude of the emf induced in a conducting loop is equal to the rate at which the magnetic flux changes with time — *Faraday's Law*

20.2. *Induction and Energy Transfer*



The figure suggests that moving a coil from a magnetic field (changing the magnetic flux) requires a force. Mechanical energy is transferred to a current dissipating thermal energy (conservation of energy).

Power (induced)

$$P = Fv = i^2R = \frac{B^2L^2v^2}{R}$$

Current (induced)

$$i = \frac{BLv}{R}$$

Electromotive Force (induced)

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d}{dt}BLx = BL\frac{dx}{dt} = BLv$$

20.3. *Inductors and Inductance*

While the basic type of a capacitor was a parallel-plate arrangement, the basic type of an inductor is a long solenoid with N loops (or a short length near the middle of a non-ideal solenoid). A current i is producing a magnetic flux Φ_B. We define the

Inductance

$$L = \frac{N\Phi_B}{i}$$

The SI unit of the inductance is the henry (H) or tesla times square-meter per ampere (H = T m^2/A).

Inductance (ideal soneloid of length l)

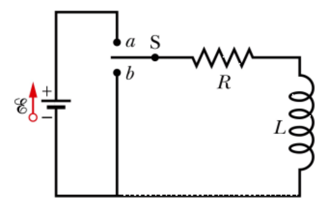
$$L = \mu_0 N^2 A \cdot l$$

An induced emf appears in any coil in which the current is changing — *Self-Induction*

Self-Induction

$$\mathcal{E}_L = -L\frac{di}{dt}$$

20.4. *LR Circuits*



The figure shows a simple LR circuit, where the switch is used to rise a current in the inductor.

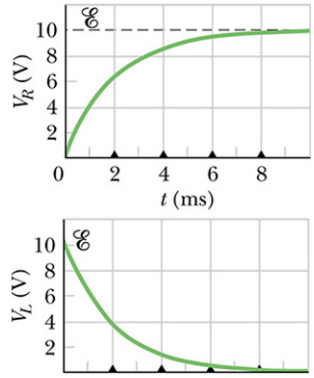
To describe the circuit after the switch is closed, we apply Kirkhoff's loop rule. The potential difference over the resistor and the inductor equals in magnitude that over the emf but having a different sign.

Kirkhoff's Loop

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$

Current (rise)

$$i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$



The plots visualize the potential differences across the resistor R and the inductance L. The curves are plotted for R = 2000 Ω, L = 4 H, and $\mathcal{E}$= 10 V. Their exponential shape is clearly observed.

An inductor initially acts like an ordinary opened wire, since it tries to oppose changes in the current. A long time later, it acts like an ordinary connecting wire.

The quotient L/R has the dimension of time. It is called the inductive time constant.

Time Constant (inductive)

$$\tau = \frac{L}{R}$$

When an inductor discharges through a resistance R, the current in the inductor decays accordingly.

Current (decay)

$$i = \frac{\mathcal{E}}{R} e^{-t/\tau} = i_0 e^{-t/\tau}$$

20.5. Energy Stored in a Magnetic Field

Energy that is delivered to the LR circuit but does not appear as thermal energy must – by the conservation-of-energy hypothesis – be stored in the magnetic field of the inductor.

Potential Magnetic Energy

$$U_B = E_{\text{pot}} = \frac{1}{2} L i^2$$

In vacuum, the energy density u (potential energy per unit volume) within a magnetic field of magnitude B is given by

Energy Density in a Magnetic Field

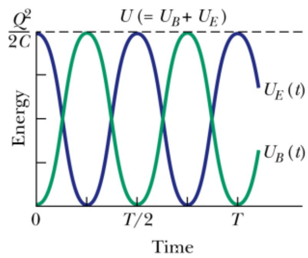
$$u = \frac{B^2}{2\mu_0}$$

20.6. LC Circuit (Electromagnetic Oscillation)

In a circuit connecting a capacitor with an inductor, neither charge, nor current, nor potential difference decay exponentially with time but they all vary sinusoidally. In analogy to the block-spring system, energy is transferred between electric and magnetic potential energy.

The table compares the energy of mechanic and electric oscillators.

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
	$v = dx/dt$		$i = dq/dt$



The plots show the stored magnetic (green) and electric (blue) energy in the LC circuit. Their sum remains constant.

LC Oscillation

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

Angular Frequency

$$\omega = \frac{1}{\sqrt{LC}}$$

Charge Oscillation

$$q = Q\cos(\omega t + \phi)$$

Current Oscillation

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t + \phi)$$

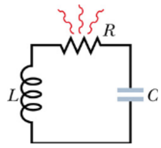
Electric Energy Oscillation

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi)$$

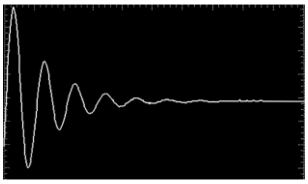
Magnetic Energy Oscillation

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi)$$

20.7. RLC DC Circuit (Damped Electromagnetic Oscillation)



The figure depicts a series of RLC circuit. As the charge contained in the circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, damping the oscillation.



The oscilloscope visualizes the damped oscillation.

Relating the electromagnetic energy U = $Li^2/2 + q^2/2C$ with the rate of transfer to thermal energy $dU/dt = -i^2R$ and substituting $i = dq/dt$ and $di/dt = d^2q/dt^2$ we yield

Damped RLC Circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

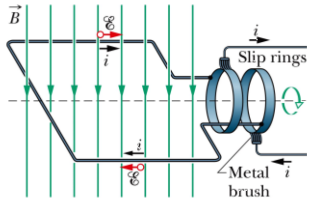
Charge (damped oscillation)

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi)$$

Angular Frequency (damped oscillation)

$$\omega' = \sqrt{\omega^2 - (R/2L)^2}$$

20.8. Alternating Current (AC)

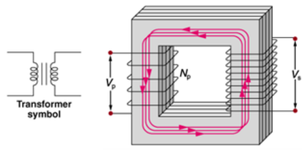


The figure shows the principle of an ideal AC emf without internal resistance. A conducting loop is rotated in an external magnetic field. Mechanical energy is transferred to electrical energy.

In practice, a conducting coil of N loops and area A is rotated with frequency ω in a magnetic field B.

AC emf (generator)

$$\mathcal{E}(t) = \mathcal{E}_m \sin(\omega t), \quad \mathcal{E}_m = NBA\omega$$



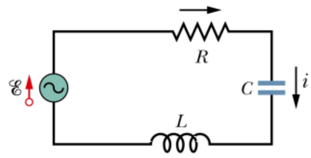
The figure demonstrates that alternating currents can be transformed if two coils wound on an iron core. Supposing 100% efficiency, the continuously changing flux is caught in the iron.

The rate of transformation of the potential difference V is determined by the primary N_p and secondary N_s number of loops.

Potential difference (secondary)

$$V_s = -N_s\frac{\Delta\Phi}{\Delta t} = V_p\frac{N_s}{N_p}$$

20.9. RLC AC Circuit (Forced Electromagnetic Oscillation)



The figure shows a RLC circuit driven from an alternating current.

The voltage across a resistor is in phase with the current. The voltage across capacitor or inductor it lacks or leads the current, respectively. We use the complex-valued impedance Z = R + jX with reactance X.

Current

$$i(t) = \frac{\mathcal{E}_m}{Z}\sin(\omega_d t - \phi)$$

Impedance (definition)

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Reactance (capacitive load)

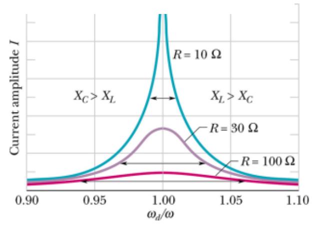
$$X_C = \frac{1}{2\pi f C}$$

Reactance (inductive load)

$$X_L = 2\pi f L$$

Phase constant

$$\tan\phi = \frac{X_L - X_C}{R}$$



The figure depicts the resonance curves for the driven RLC circuit. The horizontal arrows measure the curves' half-width. To the left, the curve is mainly capacitive, to the right. It's mainly inductive. On resonance, the driven frequency is $\omega_d = \omega$.